Behavioral Subtyping

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Course references

- S Gay, M Hole, **Subtyping for session types in the pi calculus**, Acta Informatica 42(2/3):191-225, 2005
- L Padovani, **Fair Subtyping for Open Session Types**, ICALP 2013, LNCS 7966:373-384

Outline

1. Basic notions
    - Motivation
    - Informal review of subtyping
    - Subtyping for finite session types
2. Recursive session types
    - Subtyping for recursive session types
    - Subtyping algorithm
    - Further reading
3. Fair subtyping
    - Motivation
    - A liveness-preserving subtyping
    - Characterizing fair subtyping
    - Two issues
    - Further reading

Outline

Motivations for subtyping

$$\frac{\Gamma_1 \vdash e : s \qquad s = t \qquad \Gamma_2, u : T \vdash P}{\Gamma_1 + \Gamma_2, u : \,![t]\,.T \vdash u![e].P}$$

Motivations for subtyping

$$\frac{\Gamma_1 \vdash e : s \qquad s \leqslant t \qquad \Gamma_2, u : T \vdash P}{\Gamma_1 + \Gamma_2, u : ![t].T \vdash u![e].P}$$

- relax constraints on types without compromising safety
- $\Rightarrow$ more well-typed programs

Motivations for subtyping

Can a channel with type

“send any number of messages”

be used according to the type

“send at most 3 messages”

?

- formally justify the mismatch between actual and allowed behaviors

Outline

Defining a subtype relation

$$s \leqslant t$$

Safe substitution

"it is safe to use a value of type s
where a value of type t is expected"

Set inclusion (aka "semantic subtyping", cf Castagna et al.)

$$[\![s]\!] \subseteq [\![t]\!]$$

Property preservation (cf Liskov et al.)

$$\forall \phi . (\forall x : t . \phi(x)) \Rightarrow (\forall y : s . \phi(y))$$

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$$\forall \phi . (\forall x : t . \phi(x)) \Rightarrow (\forall y : s . \phi(y))$$

Examples

Set inclusion

$$\texttt{Even} \leqslant \texttt{Int}$$

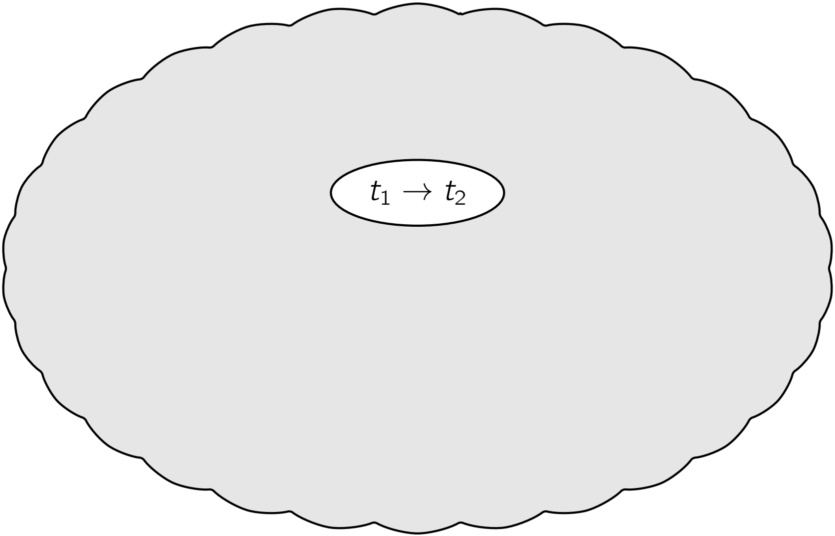
- $[\![\texttt{Even}]\!] \subseteq [\![\texttt{Int}]\!]$

Property preservation

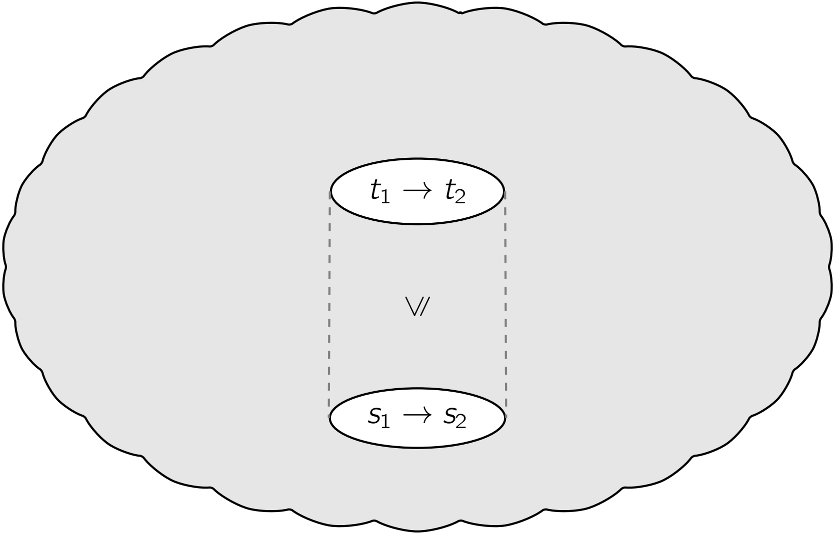
$$\{x : \texttt{Int}, y : \texttt{Int}, c : \texttt{Color}\} \leqslant \{x : \texttt{Int}, y : \texttt{Int}\}$$

- $\phi(o) =$ "o has an x field"
- $\phi(o) =$ "o has a y field"

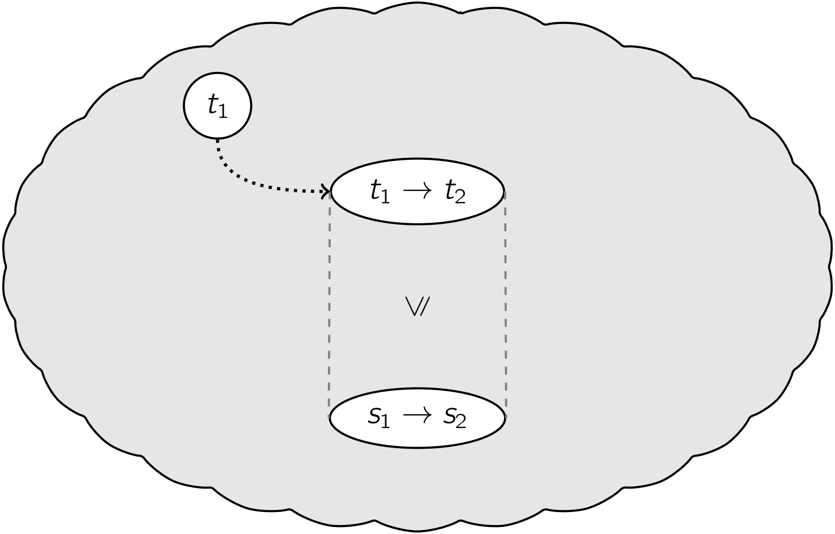
Subtyping for **function** types



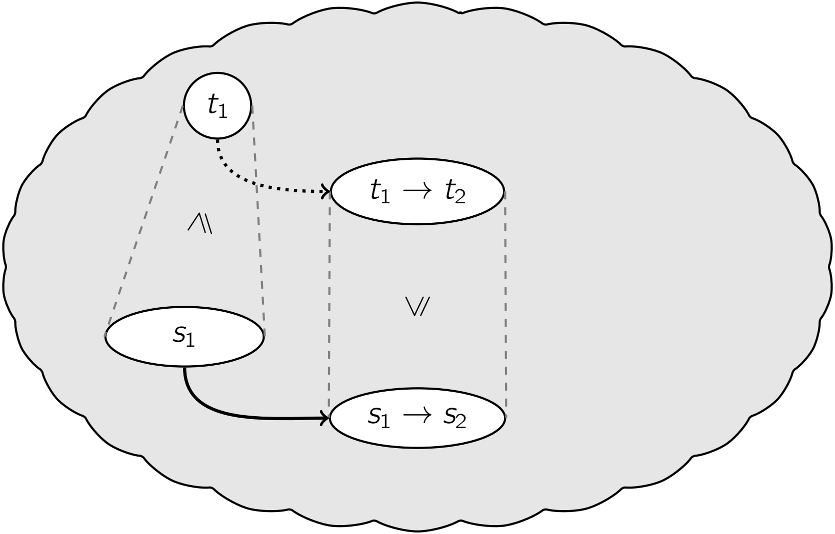
Subtyping for **function** types



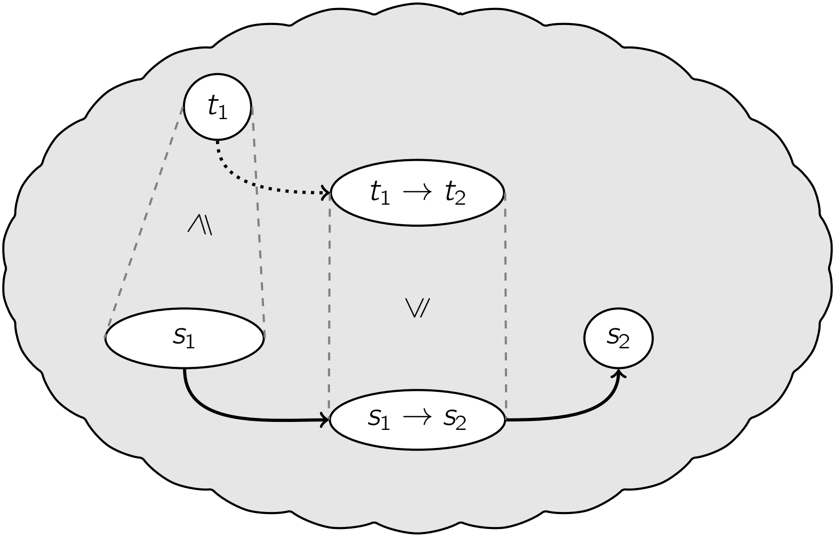
Subtyping for **function** types



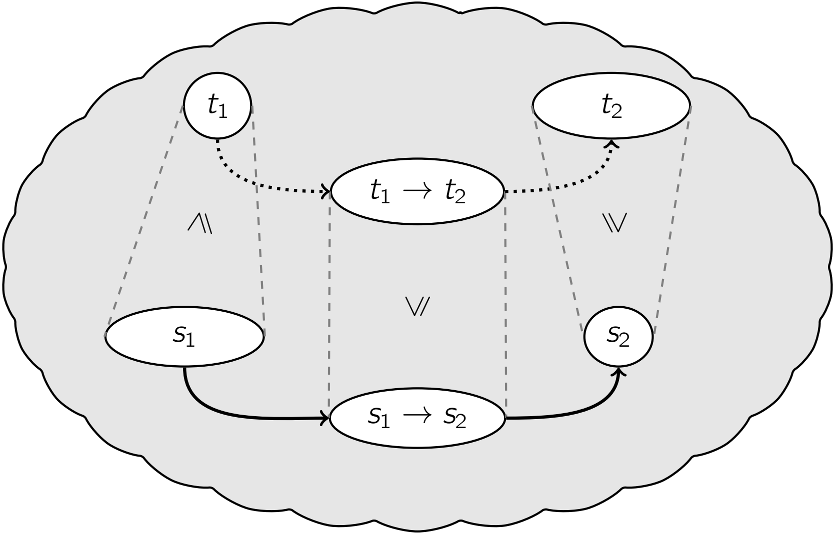
Subtyping for **function** types



Subtyping for **function** types



Subtyping for **function** types



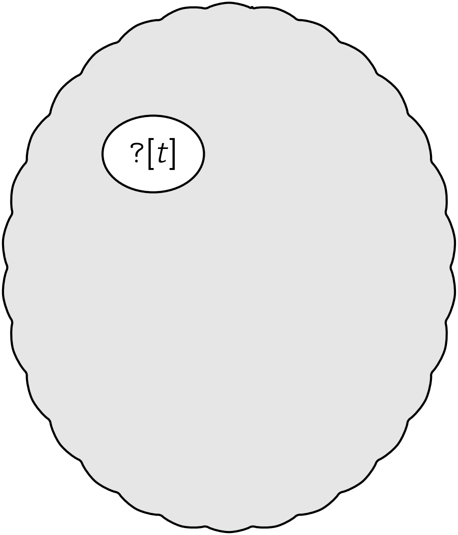
Subtyping for function types

$$\frac{t_1 \leqslant s_1 \qquad s_2 \leqslant t_2}{s_1 \to s_2 \leqslant t_1 \to t_2}$$

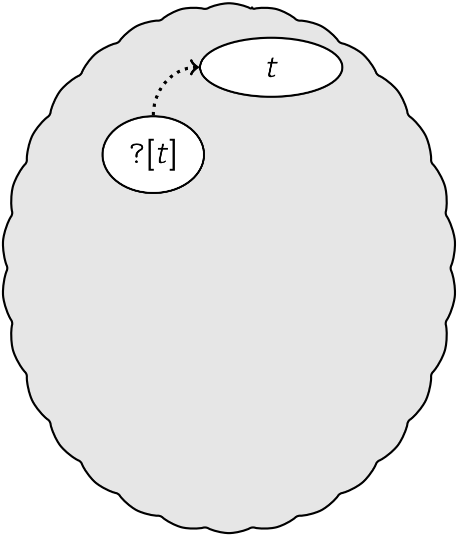
The arrow type constructor

- is **contravariant** in the domain
- is **covariant** in the codomain

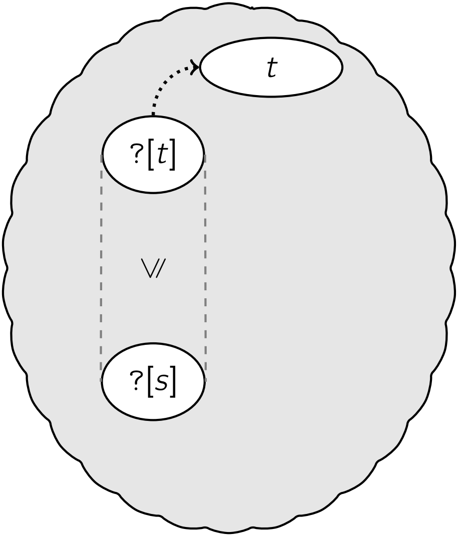
Subtyping for **input** channel types



Subtyping for **input** channel types

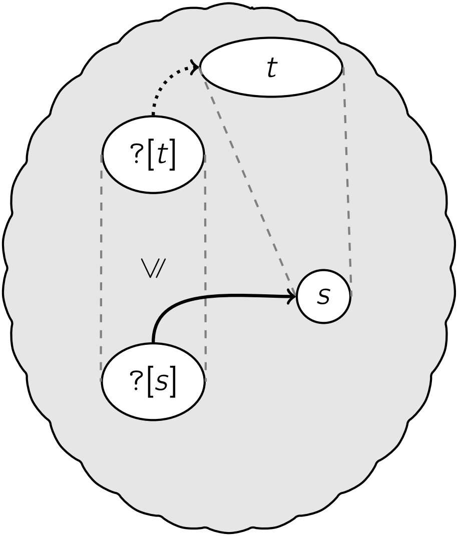


Subtyping for **input** channel types



$$\frac{}{?[s] \leqslant ?[t]}$$

Subtyping for **input** channel types



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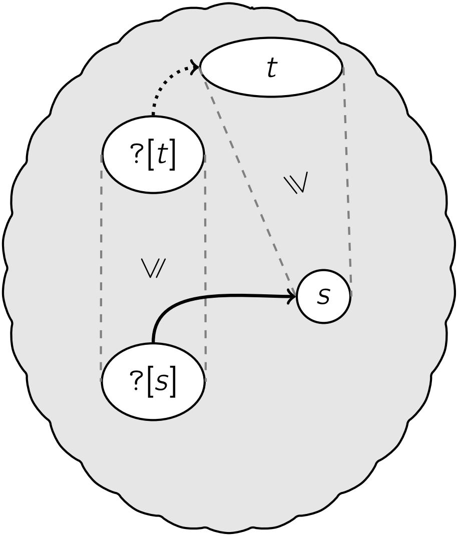
Subtyping for **input** channel types

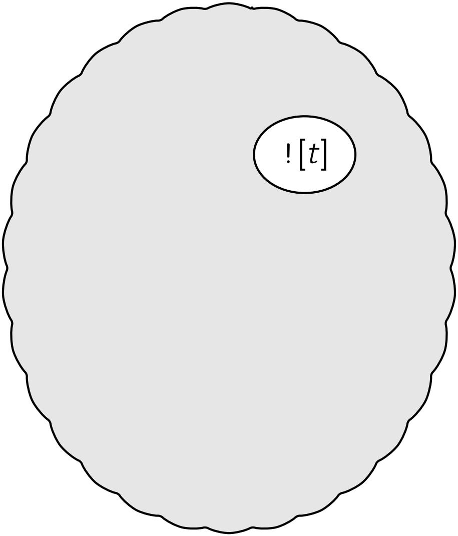


$$\frac{s \leqslant t}{?[s] \leqslant ?[t]}$$

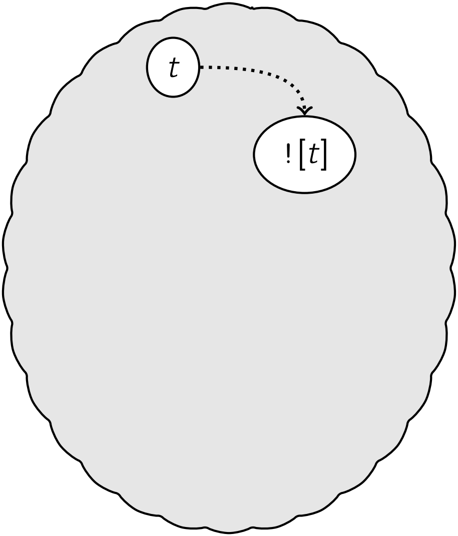
$?[\mathtt{Int}] \leqslant ?[\mathtt{Real}]$

input is covariant

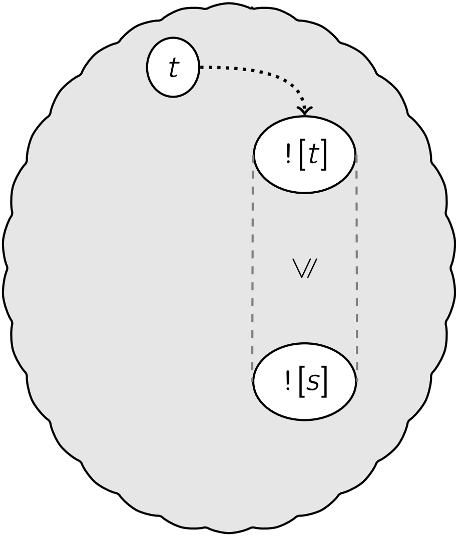
Subtyping for **output** channel types



Subtyping for **output** channel types

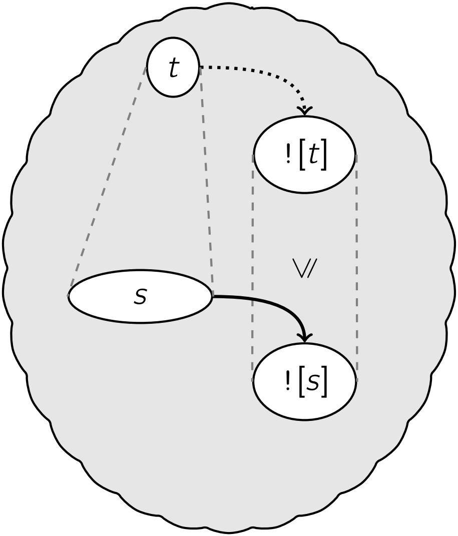


Subtyping for **output** channel types



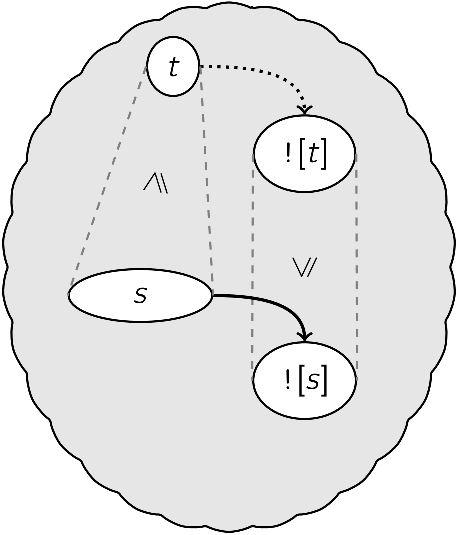
$$\overline{\;![s] \leqslant ![t]\;}$$

Subtyping for **output** channel types



$$\frac{}{![s] \leqslant ![t]}$$

Subtyping for **output** channel types



$$\frac{t \leqslant s}{![s] \leqslant ![t]}$$

$![\texttt{Real}] \leqslant ![\texttt{Int}]$

output is contravariant

Outline

Session types: syntax

T	::=		**session type**		t	::=		**type**
		`end`	(termination)				`Int`	(integer)
	|	$\texttt{?}[t]\,.\,T$	(input)			|	`Real`	(real)
	|	$\texttt{!}[t]\,.\,T$	(output)			|	T	(channel)
	|	$\&\{l_i : T_i\}_{i\in I}$	(branch)					
	|	$\oplus\{l_i : T_i\}_{i\in I}$	(choice)					

In branches and choices

- I non-empty and finite
- $l_i = l_j$ implies $i = j$

Session types: informal semantics

`end`	no operation allowed
$?[t].T$	receive a message of type t then behave according to T
$![t].T$	send a message of type t then behave according to T
$\&\{l_i : T_i\}_{i\in I}$	wait for one of the labels l_k from the set $\{l_i \mid i \in I\}$ then behave according to T_k
$\oplus\{l_i : T_i\}_{i\in I}$	choose and send a label l_k from the set $\{l_i \mid i \in I\}$ then behave according to T_k

Example

Player

$$\oplus \left\{ \begin{array}{l} \texttt{play} : \texttt{!}[\texttt{Real}] . \& \left\{ \begin{array}{l} \texttt{win} : \texttt{end} \\ \texttt{loss} : \texttt{end} \end{array} \right\} \\ \texttt{quit} : \texttt{end} \end{array} \right\}$$

Gaming service

$$\& \left\{ \begin{array}{l} \texttt{play} : \texttt{?}[\texttt{Real}] . \oplus \left\{ \begin{array}{l} \texttt{win} : \texttt{end} \\ \texttt{loss} : \texttt{end} \end{array} \right\} \\ \texttt{quit} : \texttt{end} \end{array} \right\}$$

Session types: subtype relation

[s-int-int]
$$\mathtt{Int} \leqslant \mathtt{Int}$$

[s-real-real]
$$\mathtt{Real} \leqslant \mathtt{Real}$$

[s-int-real]
$$\mathtt{Int} \leqslant \mathtt{Real}$$

[s-end]
$$\mathtt{end} \leqslant \mathtt{end}$$

[s-in]
$$\frac{s \leqslant t \qquad S \leqslant T}{?[s].S \leqslant ?[t].T}$$

[s-out]
$$\frac{t \leqslant s \qquad S \leqslant T}{![s].S \leqslant ![t].T}$$

[s-branch]
$$\frac{I \subseteq J \qquad S_i \leqslant T_i \ ^{i \in I}}{\&\{l_i : S_i\}_{i \in I} \leqslant \&\{l_j : T_j\}_{j \in J}}$$

[s-choice]
$$\frac{J \subseteq I \qquad S_j \leqslant T_j \ ^{j \in I}}{\oplus\{l_i : S_i\}_{i \in I} \leqslant \oplus\{l_j : T_j\}_{j \in J}}$$

Examples

$$\oplus \left\{ \begin{matrix} \texttt{play} : ![\text{Real}] . \& \left\{ \begin{matrix} \texttt{win} : \texttt{end} \\ \texttt{loss} : \texttt{end} \end{matrix} \right\} \\ \texttt{quit} : \texttt{end} \end{matrix} \right\} \leqslant \oplus \{ \texttt{quit} : \texttt{end} \}$$

$$\oplus \left\{ \begin{matrix} \texttt{play} : ![\text{Real}] . \& \left\{ \begin{matrix} \texttt{win} : \texttt{end} \\ \texttt{loss} : \texttt{end} \end{matrix} \right\} \\ \texttt{quit} : \texttt{end} \end{matrix} \right\} \leqslant \oplus \left\{ \texttt{play} : ![\text{Int}] . \& \left\{ \begin{matrix} \texttt{win} : \texttt{end} \\ \texttt{loss} : \texttt{end} \\ \texttt{tie} : \texttt{end} \end{matrix} \right\} \right\}$$

Exercises

$?[![\mathtt{Int}].\mathtt{end}].\mathtt{end}$

$?[![\mathtt{Real}].\mathtt{end}].\mathtt{end}$

$![![\mathtt{Int}].\mathtt{end}].\mathtt{end}$

$![![\mathtt{Real}].\mathtt{end}].\mathtt{end}$

$$\oplus\begin{Bmatrix}\mathtt{win} : \mathtt{end} \\ \mathtt{loss} : \mathtt{end}\end{Bmatrix}$$

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$$!\left[\oplus\begin{Bmatrix}\mathtt{win} : \mathtt{end} \\ \mathtt{loss} : \mathtt{end}\end{Bmatrix}\right].\mathtt{end}$$

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$$\&\begin{Bmatrix}\mathtt{win} : ?[\mathtt{Int}].\mathtt{end} \\ \mathtt{loss} : ![\mathtt{Int}].\mathtt{end}\end{Bmatrix}$$

Exercises

$$?[!\,[\mathtt{Int}].\mathtt{end}].\mathtt{end} \qquad \geqslant \qquad ?[!\,[\mathtt{Real}].\mathtt{end}].\mathtt{end}$$

$$![!\,[\mathtt{Int}].\mathtt{end}].\mathtt{end} \qquad\qquad ![!\,[\mathtt{Real}].\mathtt{end}].\mathtt{end}$$

$$\oplus\begin{Bmatrix}\mathtt{win} : \mathtt{end} \\ \mathtt{loss} : \mathtt{end}\end{Bmatrix} \qquad\qquad \oplus\begin{Bmatrix}\mathtt{win} : \mathtt{end} \\ \mathtt{loss} : \mathtt{end} \\ \mathtt{tie} : \mathtt{end}\end{Bmatrix}$$

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Exercises

$$?[\,![\mathtt{Int}].\mathsf{end}].\mathsf{end} \quad \geqslant \quad ?[\,![\mathtt{Real}].\mathsf{end}].\mathsf{end}$$

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Exercises

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Exercises

$$?[\,![\texttt{Int}].\texttt{end}].\texttt{end} \quad \geqslant \quad ?[\,![\texttt{Real}].\texttt{end}].\texttt{end}$$

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Exercises

$$?[\,![\mathtt{Int}].\mathsf{end}].\mathsf{end} \quad\geqslant\quad ?[\,![\mathtt{Real}].\mathsf{end}].\mathsf{end}$$

$$![\,![\mathtt{Int}].\mathsf{end}].\mathsf{end} \quad\leqslant\quad ![\,![\mathtt{Real}].\mathsf{end}].\mathsf{end}$$

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$$\&\begin{Bmatrix}\mathtt{win} : ?[\mathtt{Real}].\mathsf{end} \\ \mathtt{loss} : ![\mathtt{Real}].\mathsf{end}\end{Bmatrix} \quad\nleqslant\ngeqslant\quad \&\begin{Bmatrix}\mathtt{win} : ?[\mathtt{Int}].\mathsf{end} \\ \mathtt{loss} : ![\mathtt{Int}].\mathsf{end}\end{Bmatrix}$$

Basic properties

Proposition

$\leqslant$ *is a partial order*

Proposition

$s \leqslant t$ *can be decided in linear time*

Proof.

Subtyping rules are syntax directed.
Let $|t|$ be the number of subterms in t. In a derivation for $s \leqslant t$ each subterm of s (resp. t) occurs at most once. □

Basic properties

Proposition

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Type system (excerpt)

$$\frac{}{\vdash u![v].P}\ [\text{t-outS}]$$

Type system (excerpt)

$$\frac{}{u : \,!\,[T]\,.\,T' \vdash u!\,[v].P}\ \text{[t-outS]}$$

Type system (excerpt)

$$\frac{}{v : S, u : \mathtt{!}[T]\,.\,T' \vdash u\mathtt{!}[v].P}\ \text{[t-outS]}$$

Type system (excerpt)

$$\frac{S \leqslant T}{v : S, u : \mathtt{!}[T]\,.\,T' \vdash u\mathtt{!}[v].P}\ \text{[t-outS]}$$

Type system (excerpt)

$$\frac{S \leqslant T \qquad \Gamma, u : T' \vdash P}{\Gamma, v : S, u : {!}[T]\,.\,T' \vdash u{!}[v].P}\ \text{[t-outS]}$$

Type system (excerpt)

$$\frac{S \leqslant T \qquad \Gamma, u : T' \vdash P}{\Gamma, v : S, u : ![T].T' \vdash u![v].P} \text{ [t-outS]}$$

$$\frac{}{\vdash u \triangleright \{l_j : P_j\}_{j \in J}} \text{ [t-offer]}$$

Type system (excerpt)

$$\frac{S \leqslant T \qquad \Gamma, u : T' \vdash P}{\Gamma, v : S, u : ![T].T' \vdash u![v].P} \text{ [t-outS]}$$

$$\frac{}{u : \&\{l_i : T_i\}_{i \in I} \vdash u \triangleright \{l_j : P_j\}_{j \in J}} \text{ [t-offer]}$$

Type system (excerpt)

$$\frac{S \leqslant T \qquad \Gamma, u : T' \vdash P}{\Gamma, v : S, u : \,![T]\,.\,T' \vdash u![v].P} \text{ [t-outS]}$$

$$\frac{I \subseteq J}{u : \&\{l_i : T_i\}_{i \in I} \vdash u \triangleright \{l_j : P_j\}_{j \in J}} \text{ [t-offer]}$$

Type system (excerpt)

$$\frac{S \leqslant T \qquad \Gamma, u : T' \vdash P}{\Gamma, v : S, u : {!}[T].T' \vdash u{!}[v].P} \text{ [t-outS]}$$

$$\frac{I \subseteq J \qquad \Gamma, u : T_i \vdash P_i \ ^{(i \in I)}}{\Gamma, u : \&\{l_i : T_i\}_{i \in I} \vdash u \triangleright \{l_j : P_j\}_{j \in J}} \text{ [t-offer]}$$

Type system (continued)

$$\frac{S \leqslant T \qquad \Gamma, u : T', x : T \vdash P}{\Gamma, u : ?[S]\,.\,T' \vdash u?(x).P} \text{ [t-inS]}$$

$$\frac{k \in I \qquad \Gamma, u : T_k \vdash P}{\Gamma, u : \oplus\{l_i : T_i\}_{i \in I} \vdash u \lhd l_k.P} \text{ [t-choose]}$$

The substitution lemma

Lemma (substitution)

If

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
- $S \leqslant T$

then

- $\Gamma, a : S \vdash P\{a/x\}$

The substitution lemma: proof

Hypotheses

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
- $S \leqslant T$

Thesis

- $\Gamma, a : S \vdash P\{a/x\}$

$$\frac{T \leqslant S' \qquad \Gamma', u : T' \vdash Q}{\Gamma', u : \,![S']\,.T', x : T \vdash u![x].Q} \Rightarrow \overline{\qquad\qquad\qquad\qquad}$$

The substitution lemma: proof

Hypotheses

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
- $S \leqslant T$

Thesis

- $\Gamma, a : S \vdash P\{a/x\}$

$a, x \notin \mathsf{fn}(Q)$

$$\frac{T \leqslant S' \qquad \Gamma', u : T' \vdash Q}{\Gamma', u : {!}[S'].T', x : T \vdash u{!}[x].Q} \Rightarrow \overline{\qquad\qquad\qquad}$$

The substitution lemma: proof

Hypotheses

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
- $S \leqslant T$

Thesis

- $\Gamma, a : S \vdash P\{a/x\}$

$a, x \notin \mathsf{fn}(Q)$

$$\frac{T \leqslant S' \qquad \Gamma', u : T' \vdash Q}{\Gamma', u : \,![S'].T', x : T \vdash u![x].Q} \Rightarrow \frac{\Gamma', u : T' \vdash Q}{}$$

The substitution lemma: proof

Hypotheses

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
- $S \leqslant T$

Thesis

- $\Gamma, a : S \vdash P\{a/x\}$

transitivity

$$\frac{T \leqslant S' \qquad \Gamma', u : T' \vdash Q}{\Gamma', u : {!}[S'].T', x : T \vdash u![x].Q} \Rightarrow \frac{S \leqslant S' \qquad \Gamma', u : T' \vdash Q}{}$$

The substitution lemma: proof

Hypotheses

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
- $S \leqslant T$

Thesis

- $\Gamma, a : S \vdash P\{a/x\}$

$$\frac{T \leqslant S' \qquad \Gamma', u : T' \vdash Q}{\Gamma', u : \,![S']\,.T', x : T \vdash u![x].Q} \Rightarrow \frac{S \leqslant S' \qquad \Gamma', u : T' \vdash Q}{\Gamma', u : \,![S']\,.T', a : S \vdash u![a].Q}$$

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$$\frac{s \leqslant t \qquad \Gamma', x : T' \vdash Q}{\Gamma', x : {!}[t]\,.T', u : s \vdash x{!}[u].Q} \Rightarrow \frac{}{}$$

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$$\frac{s \leqslant t \qquad \Gamma', x : T' \vdash Q}{\Gamma', x : \,![t]\,.T', u : s \vdash x![u].Q} \Rightarrow \overline{\qquad\qquad}$$

$T = \,![t]\,.T'$

The substitution lemma: proof

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$$\frac{s \leqslant t \qquad \Gamma', x : T' \vdash Q}{\Gamma', x : ![t].T', u : s \vdash x![u].Q} \Rightarrow \overline{}$$

$$![s'].S' = S \leqslant T = ![t].T'$$

$$t \leqslant s' \qquad S' \leqslant T'$$

The substitution lemma: proof

Hypotheses

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
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Thesis

- $\Gamma, a : S \vdash P\{a/x\}$

ind. hyp. $(S' \leqslant T')$

$$\frac{s \leqslant t \qquad \Gamma', x : T' \vdash Q}{\Gamma', x : \,!\,[t]\,.\,T', u : s \vdash x\,!\,[u].Q} \Rightarrow \frac{\Gamma', a : S' \vdash Q\{a/x\}}{}$$

$$!\,[s']\,.S' = S \leqslant T = \,!\,[t]\,.\,T'$$

$$t \leqslant s' \qquad S' \leqslant T'$$

The substitution lemma: proof

Hypotheses

- $\Gamma, x : T \vdash P$
- $a \notin \mathsf{dom}(\Gamma)$
- $S \leqslant T$

Thesis

- $\Gamma, a : S \vdash P\{a/x\}$

transitivity

$$\frac{s \leqslant t \qquad \Gamma', x : T' \vdash Q}{\Gamma', x : {!}[t]\,.T', u : s \vdash x!\,[u].Q} \Rightarrow \frac{s \leqslant s' \qquad \Gamma', a : S' \vdash Q\{a/x\}}{}$$

$${!}[s']\,.S' = S \leqslant T = {!}[t]\,.T'$$

$$t \leqslant s' \qquad S' \leqslant T'$$

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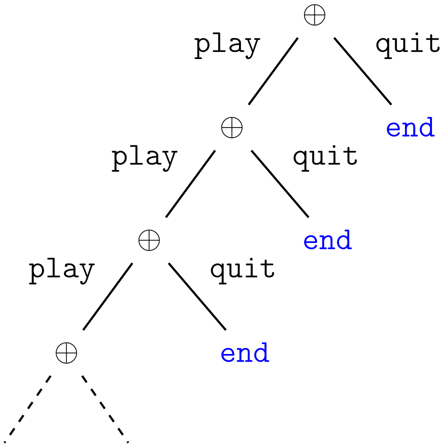
Outline

Recursive session types: motivation



- **infinite** protocols
- **finite** but **arbitrarily long** protocols

Recursive session types: syntax

- infinite supply of type variables X, Y, ...

$$
\begin{array}{rclll}
T & ::= & & & \textbf{session type} \\
 & & \vdots & & \text{as before} \\
 & | & X & & \text{(session type variable)} \\
 & | & \mu X.T & & \text{(recursion)}
\end{array}
$$

- types identified modulo renaming of bound type variables
- $\mu X_1 \cdots \mu X_n.X_1$ subterms forbidden

Example

$$\mu X.\oplus\begin{Bmatrix}\texttt{play} : X \\ \texttt{quit} : \texttt{end}\end{Bmatrix}$$

Example

$$\mu X. \oplus \begin{Bmatrix} \texttt{play} : X \\ \texttt{quit} : \texttt{end} \end{Bmatrix}$$

Example

$$\mu X.\oplus\begin{Bmatrix}\texttt{play} : X \\ \texttt{quit} : \color{blue}{\texttt{end}}\end{Bmatrix}$$

$$\wr\wr$$

$$\oplus\begin{Bmatrix}\texttt{play} : \mu X.\oplus\begin{Bmatrix}\texttt{play} : X \\ \texttt{quit} : \color{blue}{\texttt{end}}\end{Bmatrix} \\ \texttt{quit} : \color{blue}{\texttt{end}}\end{Bmatrix}$$

Example

$$\mu X.\oplus\left\{\begin{array}{l}\mathtt{play} : X\\ \mathtt{quit} : \mathtt{end}\end{array}\right\}$$

$$\wr$$

$$\oplus\left\{\begin{array}{l}\mathtt{play} : \mu X.\oplus\left\{\begin{array}{l}\mathtt{play} : X\\ \mathtt{quit} : \mathtt{end}\end{array}\right\}\\ \mathtt{quit} : \mathtt{end}\end{array}\right\}$$

$$\wr$$

$$\oplus\left\{\begin{array}{l}\mathtt{play} : \oplus\left\{\begin{array}{l}\mathtt{play} : \mu X.\oplus\left\{\begin{array}{l}\mathtt{play} : X\\ \mathtt{quit} : \mathtt{end}\end{array}\right\}\\ \mathtt{quit} : \mathtt{end}\end{array}\right\}\\ \mathtt{quit} : \mathtt{end}\end{array}\right\}$$

$$\wr$$

$$\vdots$$

On contractiveness

Intuition

$$\mu X.T$$

denotes the (possibly infinite) protocol that is solution of the equation

$$X = T$$

Problem

$\mu X.X$ that is the equation $X = X$

has infinitely many solutions, which one do we mean?

Theorem (Courcelle 1983)

Contractive (systems of) equations have ***exactly*** *one solution*

On contractiveness

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Theorem (Courcelle 1983)

Contractive (systems of) equations have **exactly** *one solution*

Subtyping recursive types: the Amber rule

$$\frac{}{\mu X.S \leqslant \mu Y.T}$$

Subtyping recursive types: the Amber rule

$$\frac{S \leqslant T}{\mu X.S \leqslant \mu Y.T}$$

Subtyping recursive types: the Amber rule

$$\frac{\Sigma, X \leqslant Y \vdash S \leqslant T}{\Sigma \vdash \mu X.S \leqslant \mu Y.T} \qquad \frac{(X \leqslant Y) \in \Sigma}{\Sigma \vdash X \leqslant Y}$$

Subtyping recursive types: the Amber rule

$$\frac{\Sigma, X \leqslant Y \vdash S \leqslant T}{\Sigma \vdash \mu X.S \leqslant \mu Y.T} \qquad \frac{(X \leqslant Y) \in \Sigma}{\Sigma \vdash X \leqslant Y}$$

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$$\frac{\Sigma, X \leqslant Y \vdash S \leqslant T}{\Sigma \vdash \mu X.S \leqslant \mu Y.T} \qquad \frac{(X \leqslant Y) \in \Sigma}{\Sigma \vdash X \leqslant Y}$$

Example

$$\frac{}{\vdash \mu X.!\,[\mathtt{Real}]\,.X \leqslant \mu Y.!\,[\mathtt{Int}]\,.Y}$$

Subtyping recursive types: the Amber rule

$$\frac{\Sigma, X \leqslant Y \vdash S \leqslant T}{\Sigma \vdash \mu X.S \leqslant \mu Y.T} \qquad \frac{(X \leqslant Y) \in \Sigma}{\Sigma \vdash X \leqslant Y}$$

Example

$$\frac{\overline{X \leqslant Y \vdash \texttt{!}[\texttt{Real}].X \leqslant \texttt{!}[\texttt{Int}].Y}}{\vdash \mu X.\texttt{!}[\texttt{Real}].X \leqslant \mu Y.\texttt{!}[\texttt{Int}].Y}$$

Subtyping recursive types: the Amber rule

$$\frac{\Sigma, X \leqslant Y \vdash S \leqslant T}{\Sigma \vdash \mu X.S \leqslant \mu Y.T} \qquad \frac{(X \leqslant Y) \in \Sigma}{\Sigma \vdash X \leqslant Y}$$

Example

$$\dfrac{\dfrac{\overline{X \leqslant Y \vdash \texttt{Int} \leqslant \texttt{Real}}}{X \leqslant Y \vdash \,!\,[\texttt{Real}]\,.X \leqslant \,!\,[\texttt{Int}]\,.Y}}{\vdash \mu X.!\,[\texttt{Real}]\,.X \leqslant \mu Y.!\,[\texttt{Int}]\,.Y}$$

Subtyping recursive types: the Amber rule

$$\frac{\Sigma, X \leqslant Y \vdash S \leqslant T}{\Sigma \vdash \mu X.S \leqslant \mu Y.T} \qquad \frac{(X \leqslant Y) \in \Sigma}{\Sigma \vdash X \leqslant Y}$$

Example

$$\frac{\dfrac{}{X \leqslant Y \vdash \texttt{Int} \leqslant \texttt{Real}} \qquad \dfrac{}{X \leqslant Y \vdash X \leqslant Y}}{\dfrac{X \leqslant Y \vdash \,!\,[\texttt{Real}]\,.X \leqslant \,!\,[\texttt{Int}]\,.Y}{\vdash \mu X.!\,[\texttt{Real}]\,.X \leqslant \mu Y.!\,[\texttt{Int}]\,.Y}}$$

Subtyping recursive types: the Amber rule

$$\frac{\Sigma, X \leqslant Y \vdash S \leqslant T}{\Sigma \vdash \mu X.S \leqslant \mu Y.T} \qquad \frac{(X \leqslant Y) \in \Sigma}{\Sigma \vdash X \leqslant Y}$$

Example

$$\dfrac{\dfrac{\overline{X \leqslant Y \vdash \texttt{Int} \leqslant \texttt{Real}} \qquad \overline{X \leqslant Y \vdash X \leqslant Y}}{X \leqslant Y \vdash \texttt{!}[\texttt{Real}].X \leqslant \texttt{!}[\texttt{Int}].Y}}{\vdash \mu X.\texttt{!}[\texttt{Real}].X \leqslant \mu Y.\texttt{!}[\texttt{Int}].Y}$$

Problem

$$\mu X.\texttt{!}[\texttt{Int}].X \not\leqslant \mu Y.\texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].Y \qquad \texttt{end} \not\leqslant \mu X.\texttt{end}$$

- some types are not related even though the protocols they denote are related (or equal)

Unfolding

A recursive session type… $\mu X.T$

…and its unfolding $T\{\mu X.T/X\}$

Proposition

As we unfold a session type, the number of topmost μ's decreases

Idea

We can unfold a session type up to its topmost non-μ constructor

$$\mathsf{unfold}(t) \stackrel{\text{def}}{=} \begin{cases} \mathsf{unfold}(T\{t/X\}) & \text{if } t = \mu X.T \\ t & \text{otherwise} \end{cases}$$

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Unfoldings vs the Amber rule

$$T \stackrel{\text{def}}{=} \mu X.\,!\,[\texttt{Int}]\,.\,!\,[\texttt{Int}]\,.X$$

$$\frac{}{T \leqslant \,!\,[\texttt{Int}]\,.\,T}$$

Unfoldings vs the Amber rule

$$T \stackrel{\text{def}}{=} \mu X.\,\texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].X$$

$$\frac{\overline{\texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].T \leqslant \texttt{!}[\texttt{Int}].T}}{T \leqslant \texttt{!}[\texttt{Int}].T}$$

Unfoldings vs the Amber rule

$$T \stackrel{\text{def}}{=} \mu X.\,\texttt{!}[\texttt{Int}]\,.\,\texttt{!}[\texttt{Int}]\,.X$$

$$\dfrac{\dfrac{\overline{\texttt{!}[\texttt{Int}]\,.\,T \leqslant T}}{\texttt{!}[\texttt{Int}]\,.\,\texttt{!}[\texttt{Int}]\,.\,T \leqslant \texttt{!}[\texttt{Int}]\,.\,T}}{T \leqslant \texttt{!}[\texttt{Int}]\,.\,T}$$

Unfoldings vs the Amber rule

$$T \stackrel{\text{def}}{=} \mu X.\,!\,[\texttt{Int}]\,.\,!\,[\texttt{Int}]\,.X$$

$$\dfrac{\dfrac{\dfrac{\overline{!\,[\texttt{Int}]\,.\,T \leqslant\, !\,[\texttt{Int}]\,.\,!\,[\texttt{Int}]\,.\,T}}{!\,[\texttt{Int}]\,.\,T \leqslant T}}{!\,[\texttt{Int}]\,.\,!\,[\texttt{Int}]\,.\,T \leqslant\, !\,[\texttt{Int}]\,.\,T}}{T \leqslant\, !\,[\texttt{Int}]\,.\,T}$$

Unfoldings vs the Amber rule

$$T \stackrel{\text{def}}{=} \mu X.\texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].X$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{T \leqslant \texttt{!}[\texttt{Int}].T}}{\texttt{!}[\texttt{Int}].T \leqslant \texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].T}}{\texttt{!}[\texttt{Int}].T \leqslant T}}{\texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].T \leqslant \texttt{!}[\texttt{Int}].T}}{T \leqslant \texttt{!}[\texttt{Int}].T}$$

Unfoldings vs the Amber rule

$$T \stackrel{\text{def}}{=} \mu X.\,\texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].X$$

$$\begin{array}{c}
\vdots \\
\hline
T \leqslant \texttt{!}[\texttt{Int}].T \\
\hline
\texttt{!}[\texttt{Int}].T \leqslant \texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].T \\
\hline
\texttt{!}[\texttt{Int}].T \leqslant T \\
\hline
\texttt{!}[\texttt{Int}].\texttt{!}[\texttt{Int}].T \leqslant \texttt{!}[\texttt{Int}].T \\
\hline
T \leqslant \texttt{!}[\texttt{Int}].T
\end{array}$$

Observations

- no unfolding yields terms with matching μ's
- we must give up the idea of using a finite derivation for relating recursive types

Type simulation

Definition (type simulation)

We say that a relation $\mathscr{R}$ is a type simulation if $s \mathrel{\mathscr{R}} t$ implies either

- $s = t = \texttt{Int}$
- $s = \texttt{Int}$ and $t = \texttt{Real}$
- $s = t = \texttt{Real}$
- $\mathsf{unfold}(s) = ?[s'].S$ and $\mathsf{unfold}(t) = ?[t'].T$ and $s' \mathrel{\mathscr{R}} t'$ and $S \mathrel{\mathscr{R}} T$
- $\mathsf{unfold}(s) = ![s'].S$ and $\mathsf{unfold}(t) = ![t'].T$ and $t' \mathrel{\mathscr{R}} s'$ and $S \mathrel{\mathscr{R}} T$
- $\mathsf{unfold}(s) = \&\{l_i : S_i\}_{i \in I}$ and $\mathsf{unfold}(t) = \&\{l_j : T_j\}_{j \in J}$ and $I \subseteq J$ and $S_i \mathrel{\mathscr{R}} T_i$ for every $i \in I$
- $\mathsf{unfold}(s) = \oplus\{l_i : S_i\}_{i \in I}$ and $\mathsf{unfold}(t) = \oplus\{l_j : T_j\}_{j \in J}$ and $J \subseteq I$ and $S_j \mathrel{\mathscr{R}} T_j$ for every $j \in J$

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- $\mathsf{unfold}(s) = \&\{l_i : S_i\}_{i \in I}$ and $\mathsf{unfold}(t) = \&\{l_j : T_j\}_{j \in J}$ and $I \subseteq J$ and $S_i \mathscr{R} T_i$ for every $i \in I$
- $\mathsf{unfold}(s) = \oplus\{l_i : S_i\}_{i \in I}$ and $\mathsf{unfold}(t) = \oplus\{l_j : T_j\}_{j \in J}$ and $J \subseteq I$ and $S_j \mathscr{R} T_j$ for every $j \in J$

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- $\mathsf{unfold}(s) = \&\{l_i : S_i\}_{i \in I}$ and $\mathsf{unfold}(t) = \&\{l_j : T_j\}_{j \in J}$ and $I \subseteq J$ and $S_i \mathrel{\mathscr{R}} T_i$ for every $i \in I$
- $\mathsf{unfold}(s) = \oplus\{l_i : S_i\}_{i \in I}$ and $\mathsf{unfold}(t) = \oplus\{l_j : T_j\}_{j \in J}$ and $J \subseteq I$ and $S_j \mathrel{\mathscr{R}} T_j$ for every $j \in J$

Subtyping for recursive session types

Definition (subtyping)

Let $\leqslant$ be the largest type simulation, that is

$$\leqslant \stackrel{\text{def}}{=} \bigcup_{\mathscr{R} \text{ is a type simulation}} \mathscr{R}$$

Example

Given $T \stackrel{\text{def}}{=} \mu X.\mathtt{!}[\mathtt{Int}].\mathtt{!}[\mathtt{Int}].X$ show that $T \leqslant \mathtt{!}[\mathtt{Int}].T$

Definition (type simulation)

We say that a relation $\mathscr{R}$ is a type simulation if $s \mathrel{\mathscr{R}} t$ implies either

- $s = t = \mathtt{Int}$
- $\mathsf{unfold}(s) = \mathtt{!}[s'].S' \qquad \mathsf{unfold}(t) = \mathtt{!}[t'].T' \qquad t' \mathrel{\mathscr{R}} s' \qquad S' \mathrel{\mathscr{R}} T'$

Example

Given $T \stackrel{\text{def}}{=} \mu X.\,\texttt{!}[\texttt{Int}]\,.\,\texttt{!}[\texttt{Int}]\,.X$ show that $T \leqslant \texttt{!}[\texttt{Int}]\,.\,T$

$$\mathscr{R} \stackrel{\text{def}}{=} \{(T, \texttt{!}[\texttt{Int}]\,.\,T), (\texttt{!}[\texttt{Int}]\,.\,T, T), (\texttt{Int}, \texttt{Int})\}$$

Definition (type simulation)

We say that a relation $\mathscr{R}$ is a type simulation if $s \mathrel{\mathscr{R}} t$ implies either

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- $\mathsf{unfold}(s) = \texttt{!}[s']\,.S' \quad \mathsf{unfold}(t) = \texttt{!}[t']\,.\,T' \quad t' \mathrel{\mathscr{R}} s' \quad S' \mathrel{\mathscr{R}} T'$

Example

Given $T \stackrel{\text{def}}{=} \mu X.\,![\mathtt{Int}].\,![\mathtt{Int}].X$ show that $T \leqslant\ ![\mathtt{Int}].T$

$$\mathscr{R} \stackrel{\text{def}}{=} \{(T, ![\mathtt{Int}].T), (![\mathtt{Int}].T, T), (\mathtt{Int}, \mathtt{Int})\}$$

$s = T$
$t = ![\mathtt{Int}].T$

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$$\mathscr{R} \stackrel{\text{def}}{=} \{(T,\ ![\mathtt{Int}].T), (![\mathtt{Int}].T,\ T), (\mathtt{Int}, \mathtt{Int})\}$$

$\mathsf{unfold}(s) =\ ![\mathtt{Int}].![\mathtt{Int}].T$
$\mathsf{unfold}(t) =\ ![\mathtt{Int}].T$

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Example

Given $T \stackrel{\text{def}}{=} \mu X.\,!\,[\texttt{Int}]\,.\,!\,[\texttt{Int}]\,.X$ show that $T \leqslant\, !\,[\texttt{Int}]\,.\,T$

$$\mathscr{R} \stackrel{\text{def}}{=} \{(T,\, !\,[\texttt{Int}]\,.\,T),\, (!\,[\texttt{Int}]\,.\,T,\, T),\, (\texttt{Int}, \texttt{Int})\}$$

$s' = t' = \texttt{Int}$
$S' = \,!\,[\texttt{Int}]\,.\,T$
$T' = T$

Definition (type simulation)

We say that a relation $\mathscr{R}$ is a type simulation if $s \,\mathscr{R}\, t$ implies either

- $s = t = \texttt{Int}$
- $\text{unfold}(s) = \,!\,[s']\,.S'$ $\quad$ $\text{unfold}(t) = \,!\,[t']\,.\,T'$ $\quad$ $t' \,\mathscr{R}\, s'$ $\quad$ $S' \,\mathscr{R}\, T'$

Example

Given $T \stackrel{\text{def}}{=} \mu X.\,!\,[\mathtt{Int}]\,.\,!\,[\mathtt{Int}]\,.X$ show that $T \leqslant\, !\,[\mathtt{Int}]\,.\,T$

$$\mathscr{R} \stackrel{\text{def}}{=} \{(T,\, !\,[\mathtt{Int}]\,.\,T), (!\,[\mathtt{Int}]\,.\,T,\, T), (\mathtt{Int}, \mathtt{Int})\}$$

$s = \,!\,[\mathtt{Int}]\,.\,T$
$t = T$

Definition (type simulation)

We say that a relation $\mathscr{R}$ is a type simulation if $s \mathrel{\mathscr{R}} t$ implies either

- $s = t = \mathtt{Int}$
- $\mathsf{unfold}(s) = \,!\,[s']\,.S' \quad \mathsf{unfold}(t) = \,!\,[t']\,.\,T' \quad t' \mathrel{\mathscr{R}} s' \quad S' \mathrel{\mathscr{R}} T'$

Example

Given $T \stackrel{\text{def}}{=} \mu X.![\mathtt{Int}].![\mathtt{Int}].X$ show that $T \leqslant ![\mathtt{Int}].T$

$$\mathscr{R} \stackrel{\text{def}}{=} \{(T, ![\mathtt{Int}].T), (![\mathtt{Int}].T, T), (\mathtt{Int}, \mathtt{Int})\}$$

$\mathsf{unfold}(s) = ![\mathtt{Int}].T$
$\mathsf{unfold}(t) = ![\mathtt{Int}].![\mathtt{Int}].T$

Definition (type simulation)

We say that a relation $\mathscr{R}$ is a type simulation if $s \mathrel{\mathscr{R}} t$ implies either

- $s = t = \mathtt{Int}$
- $\mathsf{unfold}(s) = ![s'].S' \quad \mathsf{unfold}(t) = ![t'].T' \quad t' \mathrel{\mathscr{R}} s' \quad S' \mathrel{\mathscr{R}} T'$

Example

Given $T \stackrel{\text{def}}{=} \mu X.![\texttt{Int}].![\texttt{Int}].X$ show that $T \leqslant ![\texttt{Int}].T$

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$s' = t' = \texttt{Int}$
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$s = t = \texttt{Int}$

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More examples

$$\mu X.\oplus\begin{Bmatrix}\texttt{play} : X \\ \texttt{quit} : \texttt{end}\end{Bmatrix} \leqslant \mu Y.\oplus\left\{\texttt{play} : \oplus\begin{Bmatrix}\texttt{play} : Y \\ \texttt{quit} : \texttt{end}\end{Bmatrix}\right\}$$

$$\mu X.\oplus\begin{Bmatrix}\texttt{play} : X \\ \texttt{quit} : \texttt{end}\end{Bmatrix} \leqslant \mu Y.\oplus\{\texttt{play} : Y\}$$

$$\mu X.\oplus\begin{Bmatrix}\texttt{play} : X \\ \texttt{quit} : \texttt{end}\end{Bmatrix} \leqslant \oplus\{\texttt{quit} : \texttt{end}\}$$

$\leqslant$ is a pre-order (but not a partial order)

Lemma

$\leqslant$ *is transitive*

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Proof.

It is enough to show that

$$\mathscr{R} \stackrel{\text{def}}{=} \{(s,t) \mid \exists u : s \leqslant u \wedge u \leqslant t\}$$

is a type simulation.

Suppose $s \mathscr{R} t$. Then $s \leqslant u$ and $u \leqslant t$ for some type u.

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Suppose $\text{unfold}(s) = ![s'].S$. From the hypothesis $s \leqslant u$ we deduce that $\text{unfold}(u) = ![u'].U$ and $u' \leqslant s'$ and $S \leqslant U$. From the hypothesis $u \leqslant t$ we deduce that $\text{unfold}(t) = ![t'].T$ and $t' \leqslant u'$ and $U \leqslant T$. Then $t' \mathscr{R} s'$ and $S \mathscr{R} T$ by definition of $\mathscr{R}$.

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Outline

Subtyping algorithm

```
subtype(s_0, t_0)
  ℛ := ∅
  𝒮 := {(s_0, t_0)}
  while 𝒮 ≠ ℛ do
    let (s, t) ∈ 𝒮 \ ℛ
    let s' = unfold(s)
    let t' = unfold(t)
    if s' = ![s''].S and t' = ![t''].T then
      𝒮 := 𝒮 ∪ {(t'', s''), (S, T)}
    else if s' = ⊕{l_i : S_i}_{i∈I} and t' = ⊕{l_j : T_j}_{j∈J} and J ⊆ I then
      𝒮 := 𝒮 ∪ {(S_j, T_j)}_{j∈J}
    else ···
    else return false
    ℛ := ℛ ∪ {(s, t)}
  return true
```

Subtyping algorithm

subtype(s_0, t_0)

$\mathscr{R} := \emptyset$ pairs that have been checked

$\mathscr{S} := \{(s_0, t_0)\}$

```
while 𝒮 ≠ ℛ do
  let (s, t) ∈ 𝒮 \ ℛ
  let s′ = unfold(s)
  let t′ = unfold(t)
  if s′ = ![s″].S and t′ = ![t″].T then
    𝒮 := 𝒮 ∪ {(t″, s″), (S, T)}
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    𝒮 := 𝒮 ∪ {(S_j, T_j)}_{j∈J}
  else ···
  else return false
  ℛ := ℛ ∪ {(s, t)}
return true
```

Subtyping algorithm

```
subtype(s_0, t_0)
  R := ∅                      ← pairs that have been checked
  S := {(s_0, t_0)}           ← pairs that must be in R if s_0 ⩽ t_0
  while S ≠ R do
    let (s, t) ∈ S \ R
    let s' = unfold(s)
    let t' = unfold(t)
    if s' = ![s''].S and t' = ![t''].T then
      S := S ∪ {(t'', s''), (S, T)}
    else if s' = ⊕{l_i : S_i}_{i∈I} and t' = ⊕{l_j : T_j}_{j∈J} and J ⊆ I then
      S := S ∪ {(S_j, T_j)}_{j∈J}
    else ···
    else return false
    R := R ∪ {(s, t)}
  return true
```

Subtyping algorithm: example

$$s_0 \stackrel{\text{def}}{=} \mu X.\oplus \begin{Bmatrix} \texttt{play} : X \\ \texttt{quit} : \texttt{end} \end{Bmatrix} \qquad t_0 \stackrel{\text{def}}{=} \mu Y.\oplus \left\{ \texttt{play} : \oplus \begin{Bmatrix} \texttt{play} : Y \\ \texttt{quit} : \texttt{end} \end{Bmatrix} \right\}$$

$$\mathscr{R}^0 = \emptyset$$

$$\mathscr{S}^0 = \{(s_0, t_0)\}$$

$$\mathscr{R}^1 = \{(s_0, t_0)\}$$

$$\mathscr{S}^1 = \left\{(s_0, t_0), \left(s_0, \oplus \begin{Bmatrix} \texttt{play} : t_0 \\ \texttt{quit} : \texttt{end} \end{Bmatrix}\right)\right\}$$

$$\mathscr{R}^2 = \left\{(s_0, t_0), \left(s_0, \oplus \begin{Bmatrix} \texttt{play} : t_0 \\ \texttt{quit} : \texttt{end} \end{Bmatrix}\right)\right\}$$

$$\mathscr{S}^2 = \left\{(s_0, t_0), \left(s_0, \oplus \begin{Bmatrix} \texttt{play} : t_0 \\ \texttt{quit} : \texttt{end} \end{Bmatrix}\right), (\texttt{end}, \texttt{end})\right\}$$

Subtyping algorithm: example

$$s_0 \stackrel{\text{def}}{=} \mu X.\oplus \begin{Bmatrix} \texttt{play} : X \\ \texttt{quit} : \texttt{end} \end{Bmatrix} \qquad t_0 \stackrel{\text{def}}{=} \mu Y.\oplus \left\{ \texttt{play} : \oplus \begin{Bmatrix} \texttt{play} : Y \\ \texttt{quit} : \texttt{end} \end{Bmatrix} \right\}$$

$$\mathscr{R}^0 = \emptyset$$

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Subtyping algorithm: correctness

subtype(s_0, t_0)

  $\mathscr{R} := \emptyset$

  $\mathscr{S} := \{(s_0, t_0)\}$

  while $\mathscr{S} \neq \mathscr{R}$ **do**

    let $(s, t) \in \mathscr{S} \setminus \mathscr{R}$

    let $s' =$ unfold(s)

    let $t' =$ unfold(t)

    if $s' = ![s''].S$ **and** $t' = ![t''].T$ **then**

      $\mathscr{S} := \mathscr{S} \cup \{(t'', s''), (S, T)\}$

    else if $s' = \oplus\{l_i : S_i\}_{i \in I}$ **and** $t' = \oplus\{l_j : T_j\}_{j \in J}$ **and** $J \subseteq I$ **then**

      $\mathscr{S} := \mathscr{S} \cup \{(S_j, T_j)\}_{j \in J}$

    else $\cdots$

    else return false

    $\mathscr{R} := \mathscr{R} \cup \{(s,$

  return true

Invariant each pair of types in $\mathscr{R}$ has been checked and the continuations are in $\mathscr{S}$

there is no type simulation that contains (s, t), let alone (s_0, t_0)

$\mathscr{R}$ is a type simulation that contains (s_0, t_0) hence $s_0 \leqslant t_0$

Subtyping algorithm: completeness

```
subtype(s₀, t₀)
  ℛ := ∅
  𝒮 := {(s₀, t₀)}
  while 𝒮 ≠ ℛ do
    let (s, t) ∈ 𝒮 \ ℛ
    let s′ = unfold(s)
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    if s′ = ![s″].S and t′ = ![t″].T then
      𝒮 := 𝒮 ∪ {(t″, s″), (S, T)}
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```

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      𝒮 := 𝒮 ∪ {(S_j, T_j)}_{j∈J}
    else ···
    else return false
    ℛ := ℛ ∪ {(s, t)}
  return true
```

will it ever terminate?

Computing the subterms of a type

$$\mathrm{Sub}(t) \stackrel{\text{def}}{=} \begin{cases} \{t\} \cup \mathrm{Sub}(s) \cup \mathrm{Sub}(T) & \text{if } t = ?[s].T \\ & \quad \text{or } t = ![s].T \\ \{t\} \cup \bigcup_{i \in I} \mathrm{Sub}(T_i) & \text{if } t = \&\{l_i : T_i\}_{i \in I} \\ & \quad \text{or } t = \oplus\{l_i : T_i\}_{i \in I} \\ \{t\} \cup \mathrm{Sub}(T)\{t/X\} & \text{if } t = \mu X.T \\ \{t\} & \text{otherwise} \end{cases}$$

$$\mathrm{Sub}(\{t_1, \ldots, t_n\}) \stackrel{\text{def}}{=} \bigcup_{1 \le i \le n} \mathrm{Sub}(t_i)$$

Properties of Sub

Proposition

Sub(t) *is* **finite** *for every* t

Proof.

Easy induction on t. □

Lemma (closure)

Let H, K *be sets of types. Then:*

- $H \subseteq \mathrm{Sub}(H)$
- $H \subseteq K$ *implies* $\mathrm{Sub}(H) \subseteq \mathrm{Sub}(K)$
- $\mathrm{Sub}(\mathrm{Sub}(H)) = \mathrm{Sub}(H)$

Idempotency of Sub

Proof.

We prove $\mathsf{Sub}(\mathsf{Sub}(t)) = \mathsf{Sub}(t)$. Using the first two properties of Sub we have $\{t\} \subseteq \mathsf{Sub}(t)$ hence $\mathsf{Sub}(t) \subseteq \mathsf{Sub}(\mathsf{Sub}(t))$.

We prove $\mathsf{Sub}(\mathsf{Sub}(t)) \subseteq \mathsf{Sub}(t)$ by induction on t.

Suppose $t = \mu X.T$. Then

$$\mathsf{Sub}(t) = \{t\} \cup \mathsf{Sub}(T)\{t/X\} \qquad \text{def. of Sub}$$

$$\begin{aligned}
\mathsf{Sub}(\mathsf{Sub}(t)) &= \mathsf{Sub}(t) \cup \mathsf{Sub}(\mathsf{Sub}(T)\{t/X\}) && \text{def.} \\
&\subseteq \mathsf{Sub}(t) \cup \mathsf{Sub}(\mathsf{Sub}(T))\{t/X\} && \text{conjecture} \\
&\subseteq \mathsf{Sub}(t) \cup \mathsf{Sub}(T)\{t/X\} && \text{ind. hyp.} \\
&= \mathsf{Sub}(t) && \text{def. of Sub}
\end{aligned}$$

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&= \mathrm{Sub}(t) && \text{def. of Sub}
\end{aligned}$$

□

Key closure lemma

Lemma

Let $\mathsf{btv}(t) \cap \mathsf{ftv}(S) = \emptyset$*. Then* $\mathsf{Sub}(t\{S/Y\}) \subseteq \mathsf{Sub}(t)\{S/Y\} \cup \mathsf{Sub}(S)$*.*

Proof.

By induction on t. Suppose $t = \mu X.T$ and $X \neq Y$. Then

$$\begin{aligned}
\mathsf{Sub}(t\{S/Y\}) &= \mathsf{Sub}(\mu X.T\{S/Y\}) \\
&= \{t\{S/Y\}\} \cup \mathsf{Sub}(T\{S/Y\})\{t\{S/Y\}/X\} \\
&\subseteq \{t\{S/Y\}\} \cup (\mathsf{Sub}(T)\{S/Y\} \cup \mathsf{Sub}(S))\{t\{S/Y\}/X\} \\
&= \{t\{S/Y\}\} \cup \mathsf{Sub}(T)\{S/Y\}\{t\{S/Y\}/X\} \cup \mathsf{Sub}(S) \\
&= \{t\{S/Y\}\} \cup \mathsf{Sub}(T)\{t/X\}\{S/Y\} \cup \mathsf{Sub}(S) \\
&= (\{t\} \cup \mathsf{Sub}(T)\{t/X\})\{S/Y\} \cup \mathsf{Sub}(S) \\
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\end{aligned}$$

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&= \{t\{S/Y\}\} \cup \mathsf{Sub}(T)\{t/X\}\{S/Y\} \cup \mathsf{Sub}(S) \\
&= (\{t\} \cup \mathsf{Sub}(T)\{t/X\})\{S/Y\} \cup \mathsf{Sub}(S) \\
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By induction on t. Suppose $t = \mu X.T$ and $X \neq Y$. Then

$$
\begin{aligned}
\mathrm{Sub}(t\{S/Y\}) &= \mathrm{Sub}(\mu X.T\{S/Y\}) \\
&= \{t\{S/Y\}\} \cup \mathrm{Sub}(T\{S/Y\})\{t\{S/Y\}/X\} \\
&\subseteq \{t\{S/Y\}\} \cup (\mathrm{Sub}(T)\{S/Y\} \cup \mathrm{Sub}(S))\{t\{S/Y\}/X\} \\
&= \{t\{S/Y\}\} \cup \mathrm{Sub}(T)\{S/Y\}\{t\{S/Y\}/X\} \cup \mathrm{Sub}(S) \\
&= \{t\{S/Y\}\} \cup \mathrm{Sub}(T)\{t/X\}\{S/Y\} \cup \mathrm{Sub}(S) \\
&= (\{t\} \cup \mathrm{Sub}(T)\{t/X\})\{S/Y\} \cup \mathrm{Sub}(S) \\
&= \mathrm{Sub}(t)\{S/Y\} \cup \mathrm{Sub}(S)
\end{aligned}
$$

□

Relation between unfold and Sub

Lemma

$\mathsf{unfold}(t) \in \mathsf{Sub}(t)$

Proof.

By induction on the number n of topmost μs in t.

- ($n = 0$) Then $\mathsf{unfold}(t) = t \in \mathsf{Sub}(t)$
- ($n > 0$) Then $t = \mu X.T$, we have

$$\begin{aligned}
\mathsf{unfold}(t) &= \mathsf{unfold}(T\{t/X\}) && \text{def. of unfold} \\
&\in \mathsf{Sub}(T\{t/X\}) && \text{contr. + ind. hyp.} \\
&\subseteq \mathsf{Sub}(T)\{t/X\} \cup \mathsf{Sub}(t) && \text{key closure lemma} \\
&= \mathsf{Sub}(t) && \text{def. of Sub}
\end{aligned}$$

□

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$$

□

Subtyping algorithm: termination

```
subtype(s₀, t₀)
  R := ∅
  S := {(s₀, t₀)}
  while S ≠ R do
    let (s, t) ∈ S \ R
    let s' = unfold(s)
    let t' = unfold(t)
    if s' = ![s''].S and t' = ![t''].T then
      S := S ∪ {(t'', s''), (S, T)}
    else if s' = ⊕{l_i : S_i}_{i∈I} and t' = ⊕{l_j : T_j}_{j∈J} and J ⊆ I then
      S := S ∪ {(S_j, T_j)}_{j∈J}
    else ···
    else return false
    R := R ∪ {(s, t)}
  return true
```

Invariant $\mathscr{S} \subseteq (\mathsf{Sub}(s_0) \cup \mathsf{Sub}(t_0))^2$

Subtyping algorithm: complexity

- Given two types s and t, let

$$n = \max\{\#\mathsf{Sub}(s), \#\mathsf{Sub}(t)\}$$

- The algorithm performs at most n^2 iterations

- Other operations have negligible costs
  (with suitable representation of sets/session types)

Homework

- Implement the subtyping algorithm for session types

"What I cannot create, I do not understand"

Richard Feynman

Outline

General references

- B Liskov, J Wing, **A behavioral notion of subtyping**
  TOPLAS 1994
    - subtyping as property preservation

- B Pierce, **Types and Programming Languages**
  MIT Press, 2002
    - Chapters 15–19 on subtyping
    - Chapters 20–21 on recursive types

- D Sangiorgi, **Introduction to bisimulation and coinduction**
  Cambridge 2012
    - coinductively defined behavioral equivalences

Subtyping for XML

- H Hosoya, J Vouillon, B Pierce, **Regular expression types for XML**, TOPLAS 2005
- A Frisch, G Castagna, V Benzaken, **Semantic subtyping: Dealing set-theoretically with function, union, intersection, and negation types**, JACM 2008
  - XML + arrow types + boolean combinators
- S Carpineti, C Laneve, L Padovani, **PiDuce - A Project for Experimenting Web Services Technologies**, Science of Computer Programming 2009
  - XML + Web service references

Subtyping for channel types

- B Pierce, D Sangiorgi, **Typing and Subtyping for Mobile Processes**, MSCS 1996
  - covariance/contravariance for input/output
- G Castagna, R De Nicola, D Varacca, **Semantic subtyping for the pi-calculus**, TCS 2008
  - semantic subtyping for channel types
- S Gay, **Bounded polymorphism in session types**, MSCS 2008

Outline

Well-typed programs. . .

From one of Philip Wadler's talks

"Well-typed programs can't go wrong"

Milner 1978

"Well-typed programs don't get stuck"

Harper; Felleisen and Wright 1994

"Well-typed programs can't be blamed"

Wadler and Findler 2008

Well-typed programs. . .

From one of Philip Wadler's talks

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"Well-typed programs can't be blamed"

Wadler and Findler 2008

But do well-typed programs do anything good at all?

Safety and liveness properties

- Lamport, **Proving the Correctness of Multiprocess Programs**, IEEE Trans. on Software Engineering, 1977

Correctness	=	safety + liveness
Safety	=	something [bad] must not happen
Liveness	=	something [good] must happen

Safety and liveness properties

- Lamport, **Proving the Correctness of Multiprocess Programs**, IEEE Trans. on Software Engineering, 1977

Correctness	=	safety + liveness
Safety	=	something [bad] must not happen
Liveness	=	something [good] must happen

Theorem (Alpern and Schneider 1984)

Each property is the intersection of a safety property and a liveness property

Running an e-commerce site

$$\mu X.\&\left\{\begin{array}{l}\texttt{AddToCart} : X \\ \texttt{CheckOut} : \texttt{end}\end{array}\right\}$$

Client 1 **Client 2** **Client 3** **Client 4**

Running an e-commerce site

$$\mu X.\&\left\{\begin{array}{l}\texttt{AddToCart} : X \\ \texttt{CheckOut} : \texttt{end}\end{array}\right\}$$

Client 1	**Client 2**	**Client 3**	**Client 4**
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$$\mu X.\& \begin{Bmatrix} \texttt{AddToCart} : X \\ \texttt{CheckOut} : \texttt{end} \end{Bmatrix}$$

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		`CheckOut`	

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AddToCart
AddToCart
CheckOut

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CheckOut

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Client 4

AddToCart

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		`CheckOut`	

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		`CheckOut`	`AddToCart`
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AddToCart		AddToCart	AddToCart
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			AddToCart
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Client 1	**Client 2**	**Client 3**	**Client 4**
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AddToCart		AddToCart	AddToCart
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		CheckOut	AddToCart
			AddToCart
			AddToCart
			AddToCart
			AddToCart
			...

What happened?

You

Client 4

$$\mu X.\& \left\{ \begin{array}{l} \texttt{AddToCart} : X \\ \texttt{CheckOut} : \texttt{end} \end{array} \right\}$$

$$\mu X.\oplus \left\{ \begin{array}{l} \texttt{AddToCart} : X \\ \texttt{CheckOut} : \texttt{end} \end{array} \right\}$$

What happened?

You **Client 4**

$$\mu X.\oplus\left\{\texttt{AddToCart} : X\right\}$$

$$\not\leqslant$$

$$\mu X.\&\begin{Bmatrix}\texttt{AddToCart} : X \\ \texttt{CheckOut} : \texttt{end}\end{Bmatrix}$$

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$$\leqslant$$

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$$\mu X.\oplus\begin{Bmatrix}\texttt{AddToCart} : X \\ \texttt{CheckOut} : \texttt{end}\end{Bmatrix}$$

- $\leqslant$ preserves safety but not (necessarily) liveness
- can we define a **liveness-preserving** subtyping relation?

Outline

Syntax

T	$::=$		**session type**
		$\texttt{end}$	(termination)
	$\mid$	$?[t].T$	(input)
	$\mid$	$![t].T$	(output)
	$\mid$	$\&\{l_i : T_i\}_{i \in I}$	(branch)
	$\mid$	$\oplus\{l_i : T_i\}_{i \in I}$	(choice)
	$\mid$	X	(session type variable)
	$\mid$	$\mu X.T$	(recursion)

- in this part of the course: first-order session types only
- extension to higher-order session types is possible

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		`end`	(termination)
	$\mid$	$?[t].T$	(input)
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- in this part of the course: first-order session types only
- extension to higher-order session types is possible

LTS for session types

Branch / external choice

$$\frac{k \in I}{\&\{l_i : T_i\}_{i \in I} \xrightarrow{?l_k} T_k}$$

Choice / internal choice

$$\frac{k \in I \qquad \#I > 1}{\oplus\{l_i : T_i\}_{i \in I} \xrightarrow{\tau} \oplus\{l_k : T_k\}} \qquad \oplus\{l : T\} \xrightarrow{!l} T$$

Recursion

$$\frac{T\{\mu X.T/X\} \xrightarrow{\ell} S}{\mu X.T \xrightarrow{\ell} S}$$

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Recursion

$$\frac{T\{\mu X.T/X\} \xrightarrow{\ell} S}{\mu X.T \xrightarrow{\ell} S}$$

Notation for the LTS

$$\begin{array}{rcll} \alpha & ::= & & \textbf{Action} \\ & & \texttt{?a} & \text{(input / receive tag)} \\ & | & \texttt{!a} & \text{(output / send tag)} \end{array}$$

$$\begin{array}{rcll} \ell & ::= & & \textbf{Label} \\ & & \alpha & \text{(visible action)} \\ & | & \tau & \text{(invisible action)} \end{array}$$

Complement of an action

$$\overline{\texttt{?a}} \stackrel{\text{def}}{=} \texttt{!a} \qquad\qquad \overline{\texttt{!a}} \stackrel{\text{def}}{=} \texttt{?a}$$

Sessions: syntax and LTS

$$
\begin{array}{rcll}
M & ::= & & \textbf{Session} \\
 & & T & \text{(participant)} \\
 & | & M \mid M & \text{(parallel composition)}
\end{array}
$$

$$\frac{M \xrightarrow{\alpha} M' \qquad N \xrightarrow{\overline{\alpha}} N}{M \mid N \xrightarrow{\tau} M' \mid N'}$$

$$\frac{M \xrightarrow{\ell} M'}{M \mid N \xrightarrow{\ell} M' \mid N} \qquad \frac{N \xrightarrow{\ell} N'}{M \mid N \xrightarrow{\ell} M \mid N'}$$

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More notation for the LTS

$\stackrel{\tau}{\Longrightarrow}$ reflexive and transitive closure of $\stackrel{\tau}{\longrightarrow}$

$\stackrel{\alpha}{\Longrightarrow}$ $\stackrel{\tau}{\Longrightarrow}\stackrel{\alpha}{\longrightarrow}\stackrel{\tau}{\Longrightarrow}$

$\stackrel{\alpha_1\cdots\alpha_n}{\Longrightarrow}$ $\stackrel{\alpha_1}{\Longrightarrow}\ \ldots\ \stackrel{\alpha_n}{\Longrightarrow}$

φ $\alpha_1 \cdots \alpha_n$ (string of visible actions)

$\overline{\varphi}$ component-wise complement of φ

Successful session

We reserve a special tag OK (not in branches) to denote success

Definition

We say that M is successful if, for every N such that

$$M \stackrel{\tau}{\Longrightarrow} N$$

there exists N' such that

$$N \stackrel{!\mathtt{OK}}{\Longrightarrow} N'$$

Example

$$\mu Y.\&\begin{Bmatrix}\mathtt{a} : Y \\ \mathtt{b} : \oplus\{\mathtt{OK} : \mathtt{end}\}\end{Bmatrix} \quad | \quad \mu X.\oplus\begin{Bmatrix}\mathtt{a} : X \\ \mathtt{b} : \mathtt{end}\end{Bmatrix}$$

More examples

$$| \quad \oplus\{\mathtt{a} : \oplus\{\mathtt{a} : \oplus\{\mathtt{b} : \mathtt{end}\}\}\}$$

$$| \quad \mathtt{end}$$

$$\mu Y.\&\begin{Bmatrix}\mathtt{a} : Y \\ \mathtt{b} : \oplus\{\mathtt{OK} : \mathtt{end}\}\end{Bmatrix} \quad | \quad \mu X.\oplus\{\mathtt{a} : X\}$$

$$| \quad \mu X.\oplus\begin{Bmatrix}\mathtt{a} : \oplus\{\mathtt{a} : X\} \\ \mathtt{b} : \mathtt{end}\end{Bmatrix}$$

$$| \quad \oplus\begin{Bmatrix}\mathtt{a} : \mu X.\oplus\{\mathtt{a} : X\} \\ \mathtt{b} : \mathtt{end}\end{Bmatrix}$$

More examples

$$| \quad \oplus\{\mathtt{a} : \oplus\{\mathtt{a} : \oplus\{\mathtt{b} : \mathtt{end}\}\}\}$$ ☺

$$| \quad \mathtt{end}$$

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$$| \quad \oplus \begin{Bmatrix} \mathtt{a} : \mu X.\oplus\{\mathtt{a} : X\} \\ \mathtt{b} : \mathtt{end} \end{Bmatrix}$$

More examples

$$\mu Y.\&\begin{cases}\mathtt{a} : Y \\ \mathtt{b} : \oplus\{\mathtt{OK} : \mathtt{end}\}\end{cases}$$

$\mid$	$\oplus\{\mathtt{a} : \oplus\{\mathtt{a} : \oplus\{\mathtt{b} : \mathtt{end}\}\}\}$	☺
$\mid$	$\mathtt{end}$	☹
$\mid$	$\mu X.\oplus\{\mathtt{a} : X\}$	
$\mid$	$\mu X.\oplus\begin{cases}\mathtt{a} : \oplus\{\mathtt{a} : X\} \\ \mathtt{b} : \mathtt{end}\end{cases}$	
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More examples

$$| \quad \oplus\{\mathtt{a} : \oplus\{\mathtt{a} : \oplus\{\mathtt{b} : \mathtt{end}\}\}\}$$ ☺

$$| \quad \mathtt{end}$$ ☹

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More examples

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More examples

$$\mid \ \oplus\{\texttt{a} : \oplus\{\texttt{a} : \oplus\{\texttt{b} : \texttt{end}\}\}\}$$ ☺

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Fair subtyping

Definition

$$S \leqslant_{\mathsf{F}} T \stackrel{\text{def}}{\iff} \forall M : (M \mid S \text{ successful}) \Rightarrow (M \mid T \text{ successful})$$

Fair subtyping

Definition

$$S \leqslant_{\mathsf{F}} T \overset{\text{def}}{\iff} \forall M : (M \mid S \text{ successful}) \Rightarrow (M \mid T \text{ successful})$$

By definition

- $\leqslant_{\mathsf{F}}$ preserves success (which is a liveness property)
- $\leqslant_{\mathsf{F}}$ is a pre-order

Examples

$$\mu X.\oplus\begin{Bmatrix}\mathtt{a} : X \\ \mathtt{b} : \mathtt{end}\end{Bmatrix} \qquad \leqslant \qquad \leqslant_{\mathsf{F}} \qquad \oplus\{\mathtt{a} : \oplus\{\mathtt{a} : \oplus\{\mathtt{b} : \mathtt{end}\}\}\}$$

$$\mu X.\oplus\begin{Bmatrix}\mathtt{a} : X \\ \mathtt{b} : \mathtt{end}\end{Bmatrix} \qquad \leqslant \qquad \not\leqslant_{\mathsf{F}} \qquad \mu X.\oplus\{\mathtt{a} : X\}$$

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One more example: the gaming service

$$\mu X.\& \left\{ \begin{array}{l} \mathtt{play} : \oplus \left\{ \begin{array}{l} \mathtt{win} : X \\ \mathtt{loss} : X \end{array} \right\} \\ \mathtt{quit} : \mathtt{end} \end{array} \right\}$$

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$$\overset{???}{\leqslant_{\mathsf{F}}}$$

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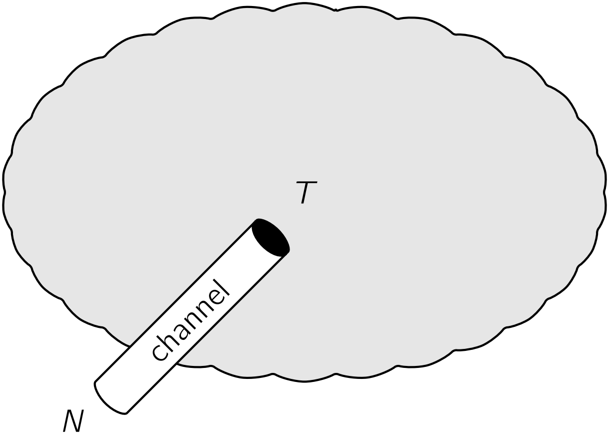
Is $\leqslant_F$ a subtyping relation?



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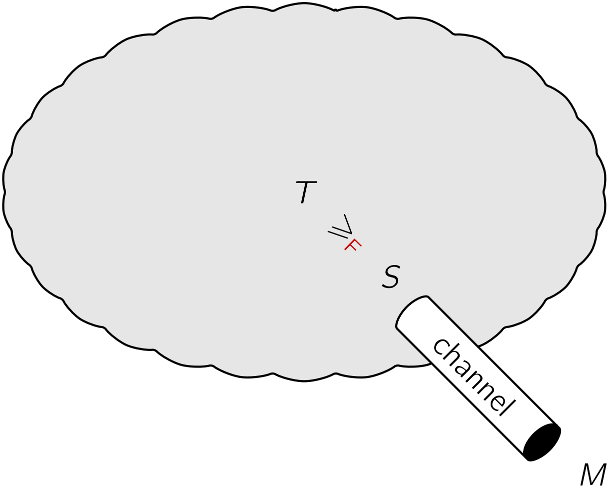
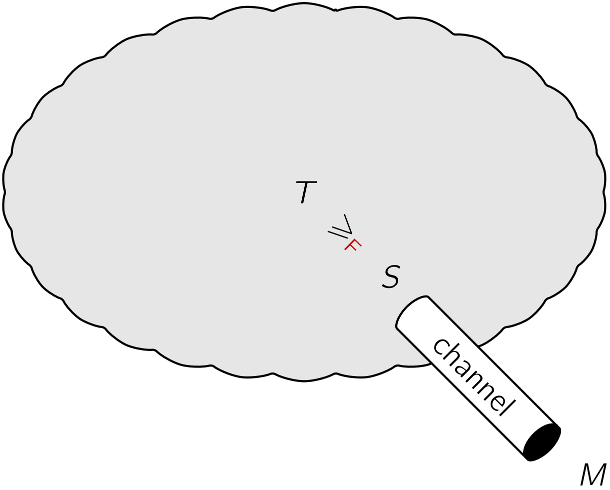


Is $\leqslant_{\mathsf{F}}$ a subtyping relation? **YES**



Digression: on the definition of "success"

Definition

We say that M is successful if, for every N such that

$$M \stackrel{\tau}{\Longrightarrow} N$$

there exists N' such that

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Definition (variation 1)

Same as original definition, but M has always the form $T_1 \mid T_2$

Digression: on the definition of “success”

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Definition (variation 2)

We say that M is successful if, for every N such that

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$$N = \texttt{end} \mid \cdots \mid \texttt{end}$$

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Definition (variation 3)

We say that M is successful if, for every N such that

$$M \stackrel{\tau}{\Longrightarrow} N$$

we have

$$N \stackrel{\tau}{\Longrightarrow} \texttt{end} \mid \cdots \mid \texttt{end}$$

Homework

1. Check whether the subtyping relation induced by variation 2 coincides with $\leqslant$
2. Show that the subtyping relation induced by variation 3 has a least element

Outline

$\leqslant_{\mathsf{F}} \subseteq \leqslant$

Definition (type simulation)

We say that a relation $\mathscr{R}$ is a type simulation if $s \mathrel{\mathscr{R}} t$ implies either

- $s = t = \texttt{Int}$
- $s = \texttt{Int}$ and $t = \texttt{Real}$
- $s = t = \texttt{Real}$
- $\mathsf{unfold}(s) = ?[s'].S$ and $\mathsf{unfold}(t) = ?[t'].T$ and $s' \mathrel{\mathscr{R}} t'$ and $S \mathrel{\mathscr{R}} T$
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- $\mathsf{unfold}(s) = \oplus\{l_i : S_i\}_{i \in I}$ and $\mathsf{unfold}(t) = \oplus\{l_j : T_j\}_{j \in J}$ and $J \subseteq I$ and $S_j \mathrel{\mathscr{R}} T_j$ for every $j \in J$

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Proof of $\leqslant_F \subseteq \leqslant$

We must show that $\leqslant_F$ is a type simulation. Suppose $S \leqslant_F T$.

- Suppose $\mathsf{unfold}(S) = \&\{l_i : S_i\}_{i \in I}$. Let $k \in I$. Then

$$\oplus\{l_k : \oplus\{\mathtt{OK} : \mathtt{end}\}\} \mid S$$

is successful. From the hypothesis $S \leqslant_F T$ we deduce

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Exercise: deduce $I = J$.

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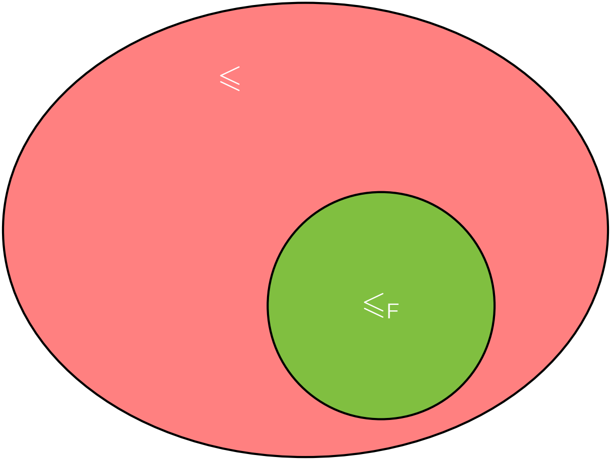
The situation so far



Notation: traces

$$\mathsf{tr}(T) \stackrel{\text{def}}{=} \{\varphi \mid T \stackrel{\varphi}{\Longrightarrow}\}$$

Examples

- $\mathsf{tr}(\mathtt{end}) = \{\varepsilon\}$
- $\mathsf{tr}(\oplus\{\mathtt{a} : \mathtt{end}, \mathtt{b} : \mathtt{end}\}) = \{\varepsilon, !\mathtt{a}, !\mathtt{b}\}$
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Notation: type continuation

Definition (continuation)

Let $T \stackrel{\alpha}{\Longrightarrow}$. The continuation of T after α is *the* session type S such that $T \stackrel{\tau}{\Longrightarrow}\stackrel{\alpha}{\longrightarrow} S$. We generalize continuations to *strings* of actions, thus

$$\begin{aligned} T(\varepsilon) &\stackrel{\text{def}}{=} T \\ T(\alpha\varphi) &\stackrel{\text{def}}{=} T(\alpha)(\varphi) \end{aligned}$$

Example

Let $T \stackrel{\text{def}}{=} \mu X.\oplus \begin{Bmatrix} \mathtt{a} : X \\ \mathtt{b} : \mathtt{end} \end{Bmatrix}$. Then

$$T(!\mathtt{b}) = T(!\mathtt{a}!\mathtt{b}) = \mathtt{end} \qquad \text{and} \qquad T(!\mathtt{a}) = T$$

Trace convergence

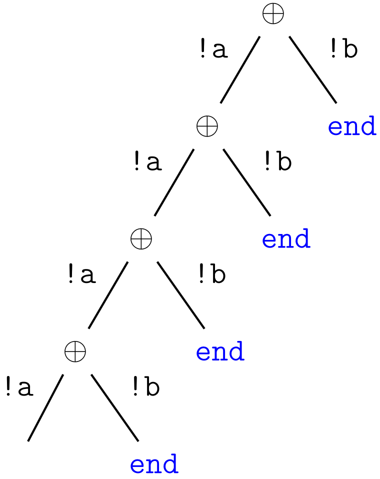
$$\frac{\forall \varphi \in \mathsf{tr}(S) \setminus \mathsf{tr}(T) : \exists \psi < \varphi, \mathtt{a} : S(\psi\,!\,\mathtt{a}) \sqsubseteq T(\psi\,!\,\mathtt{a})}{S \sqsubseteq T}$$

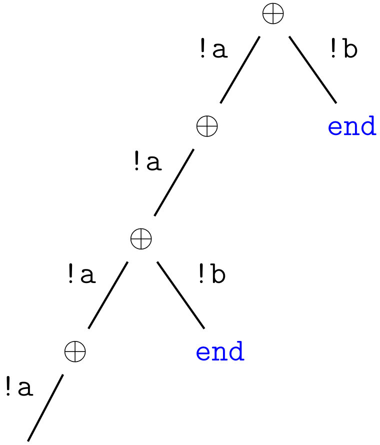
Notes

- $\sqsubseteq$ is inductively defined (least relation such that...)
- the base case is when $\mathsf{tr}(S) \subseteq \mathsf{tr}(T)$
- there is always a finite number of premises

Trace convergence: example

$$\mu X.\oplus\left\{\begin{array}{l}\mathtt{a} : X \\ \mathtt{b} : \mathtt{end}\end{array}\right\} \qquad \mu X.\oplus\left\{\begin{array}{l}\mathtt{a} : \oplus\{\mathtt{a} : X\} \\ \mathtt{b} : \mathtt{end}\end{array}\right\}$$





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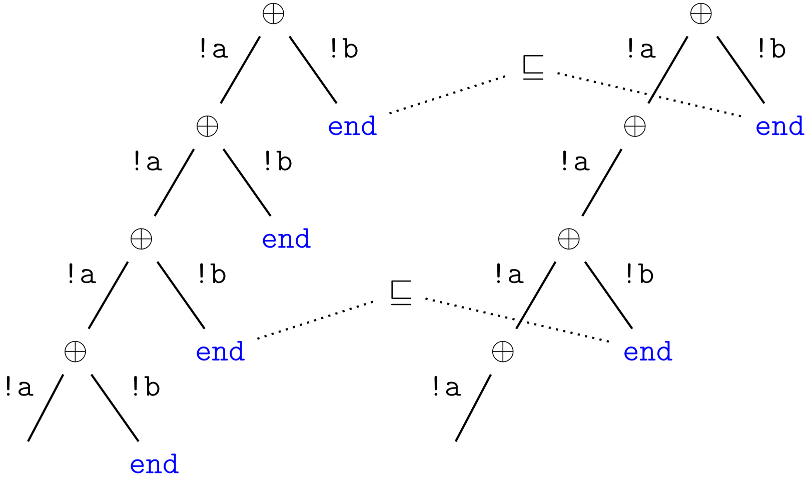

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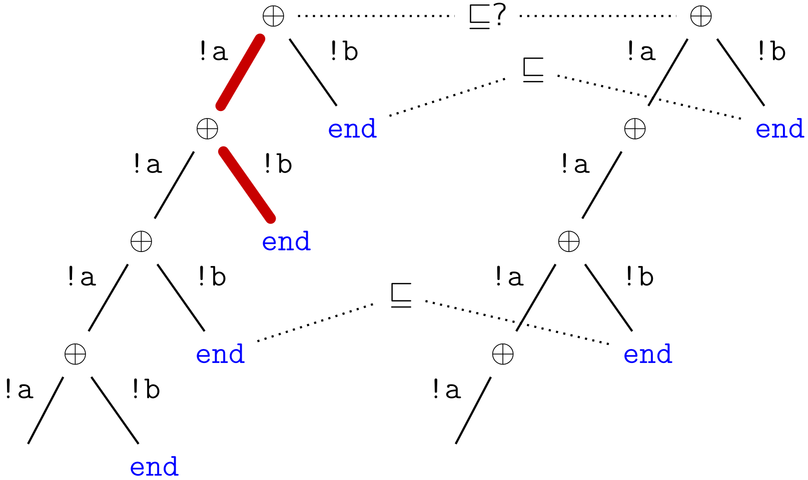

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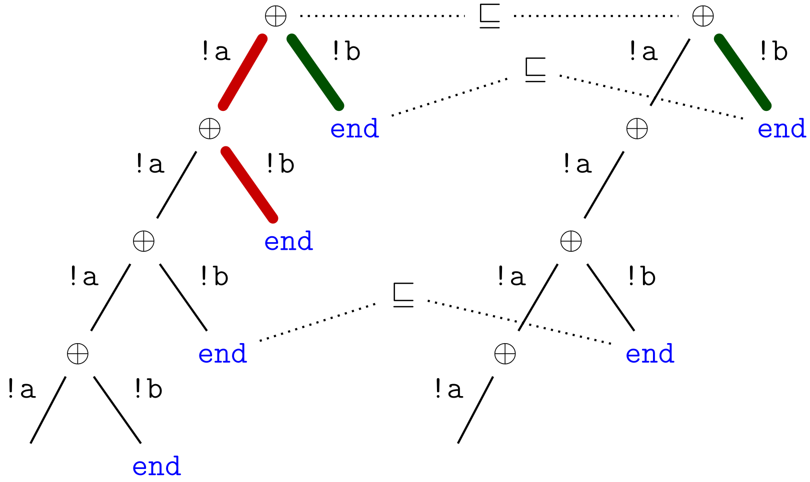
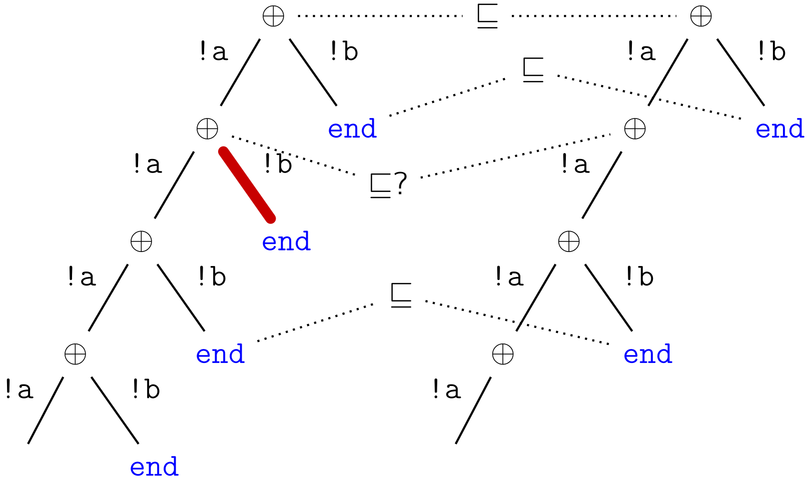


Trace convergence: example

$$\mu X.\oplus\begin{Bmatrix}\mathtt{a} : X \\ \mathtt{b} : \mathtt{end}\end{Bmatrix} \qquad \mu X.\oplus\begin{Bmatrix}\mathtt{a} : \oplus\{\mathtt{a} : X\} \\ \mathtt{b} : \mathtt{end}\end{Bmatrix}$$



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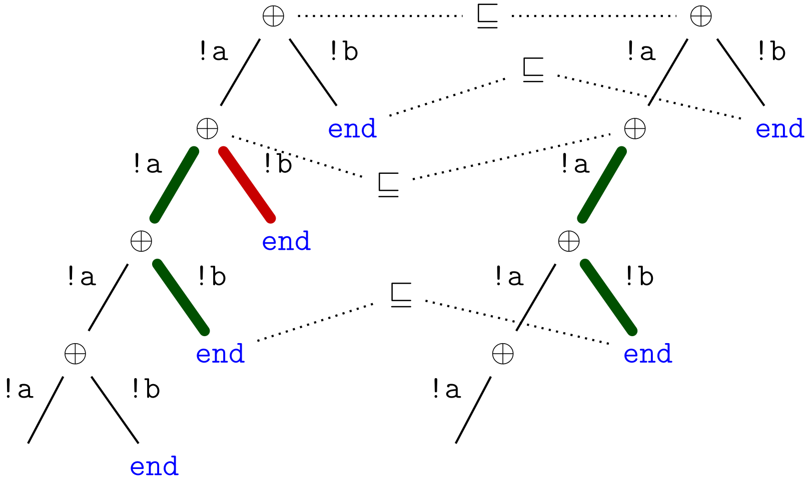

Characterization of fair subtyping

Definition (fair type simulation)

We say that a type simulation $\mathscr{R}$ is fair if $\mathscr{R} \subseteq \sqsubseteq$.

Theorem

$\leqslant_{\mathsf{F}}$ is the largest fair type simulation.

Proof plan.

Let $\preccurlyeq$ denote the largest fair type simulation. We prove the inclusions $\preccurlyeq \subseteq \leqslant_{\mathsf{F}}$ and $\leqslant_{\mathsf{F}} \subseteq \preccurlyeq$. □

$\preccurlyeq \subseteq \leqslant_{F}$

Lemma

$S \preccurlyeq T$ implies $S \leqslant_{F} T$

Proof plan.

We must prove that

$$M \mid S \qquad \text{successful}$$

implies

$$M \mid T \qquad \text{successful}$$

In particular, we must show that

$$M \mid T \xrightarrow{!\mathtt{OK}}$$

□

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Lemma (key convergence lemma)

If $S \sqsubseteq T$ and $M \mid S$ is successful, then $M \mid T \stackrel{\texttt{!OK}}{\Longrightarrow}$.

$$\frac{\forall \varphi \in \mathsf{tr}(S) \setminus \mathsf{tr}(T) : \exists \psi < \varphi, \texttt{a} : S(\psi\texttt{!a}) \sqsubseteq T(\psi\texttt{!a})}{S \sqsubseteq T}$$

Proof.

By induction on the size of the derivation of $S \sqsubseteq T$. From the hypothesis $M \mid S$ successful we have $M \stackrel{\varphi\texttt{!OK}}{\Longrightarrow}$ and $S \stackrel{\overline{\varphi}}{\Longrightarrow}$ for some φ of *minimum* length.

- (base case) $\mathsf{tr}(S) \subseteq \mathsf{tr}(T)$. We conclude $T \stackrel{\overline{\varphi}}{\Longrightarrow}$.
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If $S \sqsubseteq T$ *and* $M \mid S$ *is successful, then* $M \mid T \stackrel{\texttt{!OK}}{\Longrightarrow}$.

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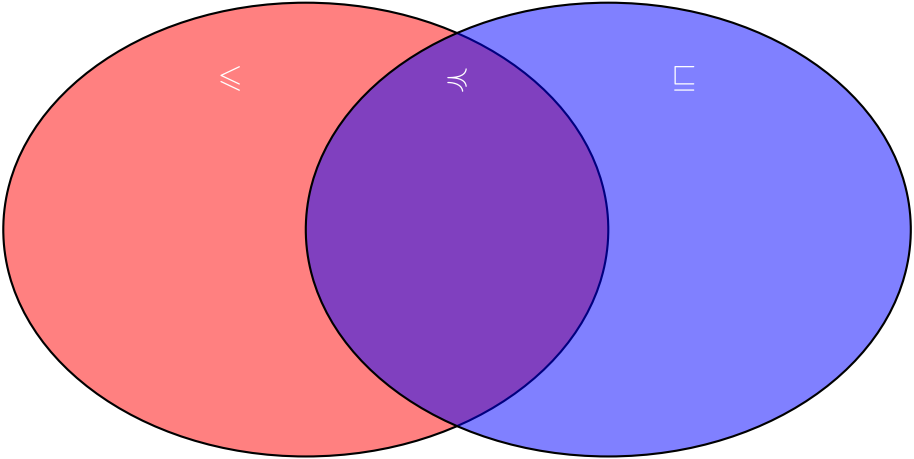
$\leqslant_{\mathsf{F}} \subseteq \preccurlyeq$



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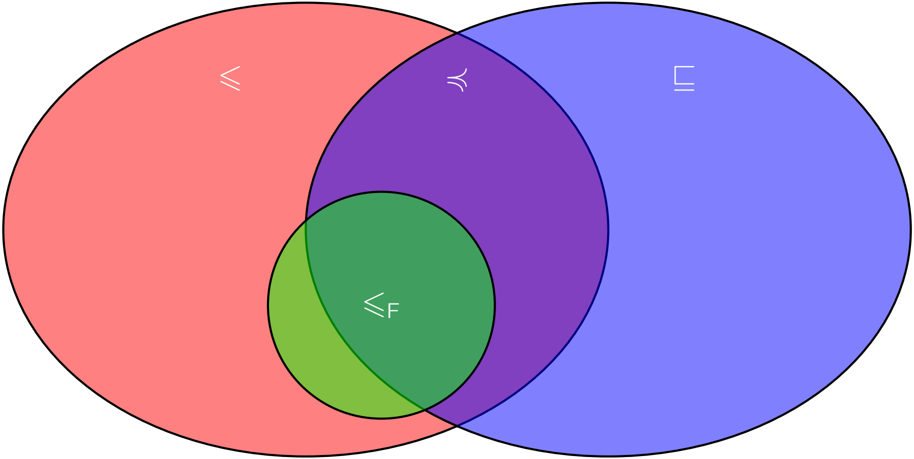

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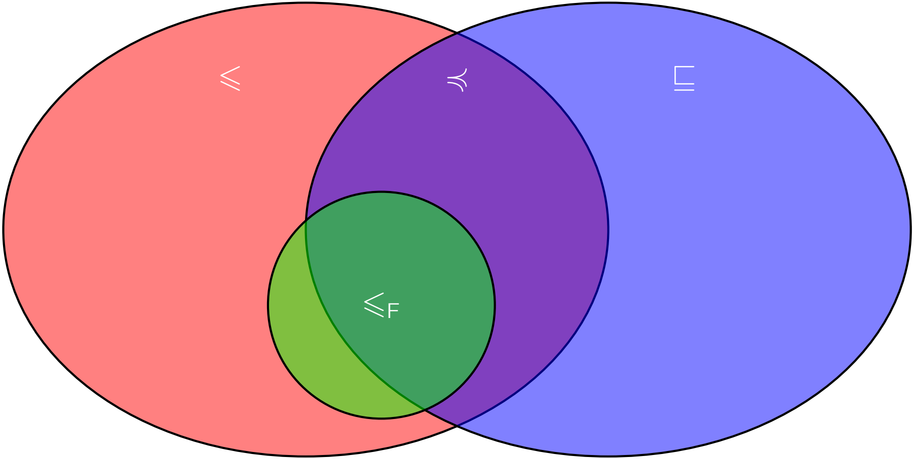

$$S \leqslant T \quad \text{and} \quad S \not\sqsubseteq T \quad \text{implies} \quad S \not\leqslant_{\mathsf{F}} T$$

Proof plan

Showing $S \not\leqslant_{\mathsf{F}} T$ amounts at finding some R such that

$$R \mid S \quad \text{is successful}$$

and

$$R \mid T \quad \text{is not}$$

under the conditions

$$S \leqslant T \quad \text{and} \quad S \not\sqsubseteq T$$

The discriminator

$$\mathscr{M}(T, S) \simeq \begin{cases} \oplus\{\mathtt{a}_i : \mathscr{M}(S_i, T_i)\}_{i \in I, S_i \not\sqsubseteq T_i} & \text{if } S \simeq \&\{\mathtt{a}_i : S_i\}_{i \in I} \\ & \quad T \simeq \&\{\mathtt{a}_j : T_j\}_{j \in J} \\ & \quad I \subseteq J \\ \&\{\mathtt{a}_j : \mathscr{M}(S_j, T_j)\}_{j \in J} & \text{if } S \simeq \oplus\{\mathtt{a}_i : S_i\}_{i \in I} \\ \quad \{\mathtt{a}_i : \oplus\{\mathtt{OK} : \mathtt{end}\}\}_{i \in I \setminus J} & \quad T \simeq \oplus\{\mathtt{a}_j : T_j\}_{j \in J} \\ & \quad J \subseteq I \end{cases}$$

- $\mathscr{M}(S, T)$ is well defined (properties of $\leqslant$ and $\not\sqsubseteq$)
- $\mathscr{M}(S, T) \mid S$ is successful (tedius)
- $\mathscr{M}(S, T) \mid T$ is unsuccessful (easy)

The discriminator

$S \not\sqsubseteq T$ implies $S_i \not\sqsubseteq T_i$ for some $i \in I$

$S \not\sqsubseteq T$ implies $S_j \not\sqsubseteq T_j$ for every $j \in J$

$$\mathscr{M}(T, S) = \begin{cases} \oplus\{\mathtt{a}_i : \mathscr{M}(S_i, T_i)\}_{i \in I, S_i \not\sqsubseteq T_i} & \text{if } S \simeq \&\{\mathtt{a}_i : S_i\}_{i \in I} \\ & \quad T \simeq \&\{\mathtt{a}_j : T_j\}_{j \in J} \\ & \quad I \subseteq J \\ \&\{\mathtt{a}_j : \mathscr{M}(S_j, T_j)\}_{j \in J} & \text{if } S \simeq \oplus\{\mathtt{a}_i : S_i\}_{i \in I} \\ \quad \{\mathtt{a}_i : \oplus\{\mathtt{OK} : \mathtt{end}\}\}_{i \in I \setminus J} & \quad T \simeq \oplus\{\mathtt{a}_j : T_j\}_{j \in J} \\ & \quad J \subseteq I \end{cases}$$

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\end{cases}
$$

receive tag $\mathtt{a}_i$ from S that cannot be sent by T

- $\mathscr{M}(S, T)$ is well defined (properties of $\leqslant$ and $\not\sqsubseteq$)
- $\mathscr{M}(S, T) \mid S$ is successful (tedius)
- $\mathscr{M}(S, T) \mid T$ is unsuccessful (easy)

Exercises

1. Show that $S \leqslant T$ implies $S \leqslant_{\mathsf{F}} T$ for all finite S, T

2. Find S, T such that $S \sqsubseteq T$ but $S \not\leqslant T$

3. Find S, T such that $S \sqsubseteq T$ but there exists $\varphi \in \mathsf{tr}(S) \cap \mathsf{tr}(T)$ such that $S(\varphi) \not\sqsubseteq T(\varphi)$

Conjecture (read: homework)

Definition

We say that M is successful if, for every N such that

$$M \stackrel{!\mathtt{REQ}}{\Longrightarrow} N$$

there exists N' such that

$$N \stackrel{!\mathtt{RESP}}{\Longrightarrow} N'$$

- See whether the subtyping relation induced by `REQ-RESP` success coincides with $\leqslant_{\mathsf{F}}$

Outline

1. Basic notions
   - Motivation
   - Informal review of subtyping
   - Subtyping for finite session types
2. Recursive session types
   - Subtyping for recursive session types
   - Subtyping algorithm
   - Further reading
3. Fair subtyping
   - Motivation
   - A liveness-preserving subtyping
   - Characterizing fair subtyping
   - Two issues
   - Further reading

Issue 1: higher-order session types

T	$::=$		**session type**
		$\texttt{end}$	(termination)
	$\mid$	$\texttt{?}[t]\texttt{.}T$	(input)
	$\mid$	$\texttt{!}[t]\texttt{.}T$	(output)
	$\mid$	$\&\{l_i : T_i\}_{i \in I}$	(branch)
	$\mid$	$\oplus\{l_i : T_i\}_{i \in I}$	(choice)
	$\mid$	X	(session type variable)
	$\mid$	$\mu X.T$	(recursion)

LTS for higher-order session types

From tags. . .

$$\frac{M \xrightarrow{\texttt{!a}} M' \qquad N \xrightarrow{\texttt{?a}} N}{M \mid N \xrightarrow{\tau} M' \mid N'}$$

LTS for higher-order session types

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From tags...

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$$\frac{M \xrightarrow{!s} M' \qquad N \xrightarrow{?t} N}{M \mid N \xrightarrow{\tau} M' \mid N'} \quad s \leqslant_{\mathsf{F}} t$$

Problem

- the LTS is used for defining $\leqslant_{\mathsf{F}}$
- $\leqslant_{\mathsf{F}}$ is used for defining the LTS

Parametric LTS and derived notions

$$\frac{M \xrightarrow{!s}_{\mathscr{S}} M' \qquad N \xrightarrow{?t}_{\mathscr{S}} N}{M \mid N \xrightarrow{\tau}_{\mathscr{S}} M' \mid N'} \quad s \,\mathscr{S}\, t$$

- $\mathscr{S}$-successful session
- $\mathscr{S}$-success induces an $\mathscr{S}$-subtyping $\mathbf{F}(\mathscr{S})$
- show that $\mathbf{F}$ has a largest fixpoint
  - **Alert!** When $\mathscr{S} \subseteq \mathscr{R}$, the number of $\mathscr{R}$-successful sessions is larger than the number of $\mathscr{S}$-successful sessions, so in principle $\mathbf{F}(\mathscr{R})$ could be smaller than or unrelated to $\mathbf{F}(\mathscr{S})$
- define $\leqslant_{\mathsf{F}}$ as the largest fixpoint of $\mathbf{F}$

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Issue 2: pre-congruence properties

$$S \leqslant_{\mathsf{F}} T \qquad \stackrel{?}{\Rightarrow} \qquad \mathscr{C}[S] \leqslant_{\mathsf{F}} \mathscr{C}[T]$$

- makes sense only if the hole in $\mathscr{C}$ is in covariant position (which is *always* the case for first-order session types)
- trivially satisfied if S and T are closed
- what about *open* session types?

The naive extension is (obviously) unsound

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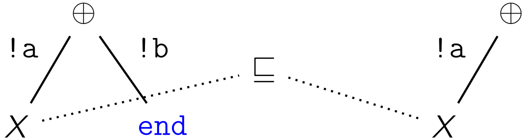

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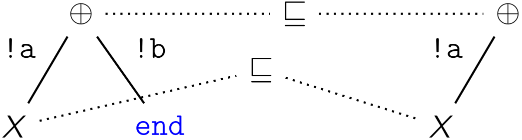
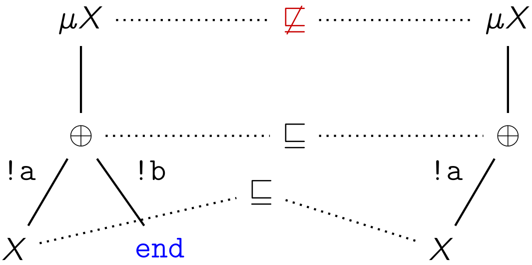
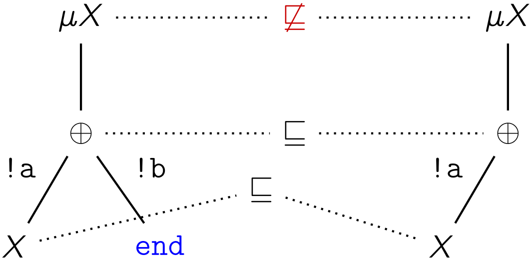


The naive extension is (obviously) unsound



The naive extension is (obviously) unsound



Convergence is context-dependent

$$\frac{X \in U}{X \sqsubseteq_U X}$$

Axioms for (iso-recursive) fair subtyping

complete axiomatization

[f-end]

$$\texttt{end} \leqslant_{\mathsf{F}} \texttt{end}$$

[f-var]

$$X \leqslant_{\mathsf{F}} X$$

[f-rec]

$$\frac{T \leqslant_{\mathsf{F}} S \qquad T \sqsubseteq_{\{X\}} S}{\mu X.T \leqslant_{\mathsf{F}} \mu X.S}$$

[f-branch]

$$\frac{\forall i \in I : S_i \leqslant_{\mathsf{F}} T_i}{\&\{\mathtt{a}_i : S_i\}_{i \in I} \leqslant_{\mathsf{F}} \&\{\mathtt{a}_i : T_i\}_{i \in I}}$$

[f-choice]

$$\frac{\forall i \in I : T_i \leqslant_{\mathsf{F}} S_i}{\oplus\{\mathtt{a}_i : S_i\}_{i \in I} \leqslant_{\mathsf{F}} \oplus\{\mathtt{a}_j : T_j\}_{j \in I \cup J}}$$

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Proposition

There is a complete algorithm for fair subtyping that runs in $O(n^4)$

Outline

Testing equivalences

- R De Nicola and M Hennessy, **Testing equivalences for processes**, TCS 1984
- M Hennessy, **Algebraic Theory of Processes**
  MIT Press 1988
  - testing equivalences for CCS processes
- D Sangiorgi, **Introduction to bisimulation and coinduction**
  Cambridge 2012
  - survey on testing equivalences

Fair testing and liveness-preserving refinements

- V Natarajan and R Cleaveland, **Divergence and fair testing**
  ICALP 1995
  - pre-congruence issues not investigated

- A Rensink and W Vogler, **Fair testing**
  Information and Computation 2007
  - more general process algebra
  - trace equivalence
  - no trace/axiomatic characterization
  - exponential decision algorithm

Models of higher-order session types

- L Padovani, **Fair Subtyping for Multi-Party Session Types**, MSCS (to appear, but on my homepage)

- G Bernardi and M Hennessy, **Using higher-order contracts to model session types**, CONCUR 2014 (to appear, but on arXiv)

Type systems for liveness properties

- N Kobayashi, **A type system for lock-free processes**, Information and Computation 2002
- S Debois, T Hildebrandt, T Slaats, N Yoshida, **Type Checking Liveness for Collaborative Processes with Bounded and Unbounded Recursion**, FORTE 2014
- L Padovani, **Deadlock and lock freedom in the linear π-calculus**, CSL-LICS 2014

  (later this week)
- . . .